THE ARBELOS

by

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I. INTRODUCTION

A. HISTORY

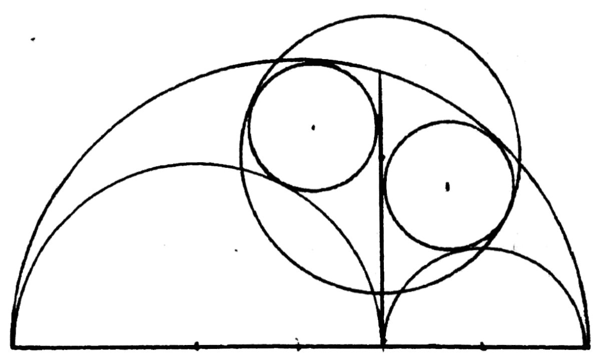
Among the works of Archimedes is the "Book of Lemmas" which contains five propositions concerning a figure known, because of its shape, as the Arbelos (ἄρβηλος) or Shoemaker's Knife. Although the Greek text seems to have been lost, there is an Arabic version made in the ninth century by the geometrician Thabit-Ben-Corrah and annotated by Almochtasso-abil-Hasan. This version was translated into Latin by Graeves and published in London in 1657 with the notes of Samuel Forster and again, by Abraham Ecchellensis, in Florence in 1661, with the notes of Borelli.

The absence of all preliminary propositions, a style different from that of other works of Archimedes, and the fact that Archimedes' name is quoted in it more than once, lead some to believe that "The Lemmas" is not entirely his work but a collection of scattered propositions. It is, however, regarded as possible, if not probable, that the theorems among them relating to the Arbelos may be due to Archimedes.

Other theorems occur in the fourth book of Pappus's "Mathematical Collection." In this work the figure similar to the Arbelos and the principle propositions respecting it are said by Pappus to be "ancient."

The principle propositions of the book of "The Lemmas" are concerned with properties of certain figures limited by semi-circles similar to the "lunules" of Hippocrates of Chio. One figure is in the shape of a knife of a thin piece of metal, called the Arbelos and another is a variation of this figure called the Salinon. These figures are in many of the geometries and famous names of history such as Viète, Descartes, Fermat, and Newton are attached. Very little was added to the original works until recent geometers applied the method of inversion to these figures.

The problem was not entirely forgotten in the meantime. During the first half of the nineteenth century several articles appeared in "The Lady's and Gentleman's Diary" concerning the properties of the Arbelos. In Leybourn's "Mathematical Repository" in 1831 question 531 by N.Y. was as follows: "If on the segments AB, Ba of the diameter of a semi-circle, the semi-circle ASB, asB be constructed, and the ordinate BQ



drawn; it is required, first, to inscribe a circle in each of the spaces AQBSA, aQBsa; secondly, to show that these two circles are equal; and thirdly, to prove that the least circle circumscribing the two equal circles, is equal to the space included between the arcs of the three semi-circles. [Solutions to these questions must come to hand (post paid) by the first day of December, 1831.]" A solution of question 531 by John Baines, Thornhill, near Wakefield is printed. At the end of the solution is the statement, "solutions were also received from Messrs. Godward, Laws and Thompson." Mr. Godward's address is given as Wakefield and Mr. T. Thompson's as Newcastle-on-Tyne. The printed solution consists of constructing the two circles in the Arbelos and proving they are equal.

Thomas Stephens Davies of Bath also contributed an article to Leybourn's Mathematical Repository concerning the Arbelos which he introduces thus:

"Some years ago my attention was directed to the figure which the ancient Greek geometers denominiated the Arbelos, or 'Gobbler's Knife;' and I was led to observe many curious properties of it, which, though they were probably known to the Greeks, are not yet recorded in any of their writings which are now known to exist. I had thought of making a distinct little treatise on the figure and the very little that is known of its history; including not only the properties I had already obtained, but such other as a renewed study of the figure might disclose to me. On this account, I allowed the time of furnishing a solution to the Respository Question to pass by. However, as other occupations have compelled me to defer it so long, that there would not be time to enter upon investigations in time for the present number, I have thought it better to furnish a few extracts from my notes, that bear a more immedi ate reference to the class of questions amongst which the proposer seems to have amused himself. If the Editor of the Respository can find me room at the end of the solution sheet for these, he will confer one little addition to the many great obligations which his kindness had laid me under.

It is possible I may resume the subject more in detail at a time of more leisure than I can now command."

He then gives some of the theorems which appear later in this thesis. He concludes his article with:

"The reader, who is familiar with the works of the ancient geometers, will recognize in this latter part, the substance of the 14-18 th propositions of the fourth book of Pappus. The subject is also treated very elegantly by the late lamented Professor Leslie, in the second book of his Geometrical Analysis. The method here pursued differs in some respects from both of these authors; whilst in others there is an almost perfect identity between all three.

In a future paper I purpose to give a more general theorem which shall contain this as a particular case, and which at the same time opens the road to a totally new series of properties of the figure."

J. S. Mackay presented an article on "The Shoemaker's Knife" which was published in the "Proceedings of the Edinburgh Mathematical Society" in 1884. The purpose of his paper was "to collect together the principle and simplest properties of the figure, and to demonstrate them in a uniform manner."

Thomas Muir also published a paper in "The Proceedings of the Edinburgh Mathematical Society" in 1885 concerning "Theorems Connected with Three Mutually Tangent Circles."

Inversion was first applied to the Arbelos by a Frenchman, Cochez, in an article published in 1877 in the "Journal de Bourget." M. d'Ocagne in an article in "l'Enseigment Mathesis" in 1934 brings out several new properties of the figure by inversion. R. Goormaghtigh, a contemporary Belgian mathematician, contributed some work on variations of the problem.

Most of the work of recent times has been contributed by another Frenchman, M. V. Thebault. He has done a great deal of work on the figure with inversion. Most of his theorems are in another section of this thesis.

B. PURPOSE

It is the purpose of this paper, first, to review the properties of the arbelos which were discovered by earlier workers who used the method of proof by demonstration; second, to present the work of recent French and Belgian mathematicians who have used a more powerful tool, the method of inversion; and third, to present some properties of the geometrical figure formed by drawing on one side of a line all possible half-circles whose diameters are the segments formed by the four points of an harmonic set, and of the volume and surface of the solid obtained by revolving this area one-half revolution about its bounding diameter.

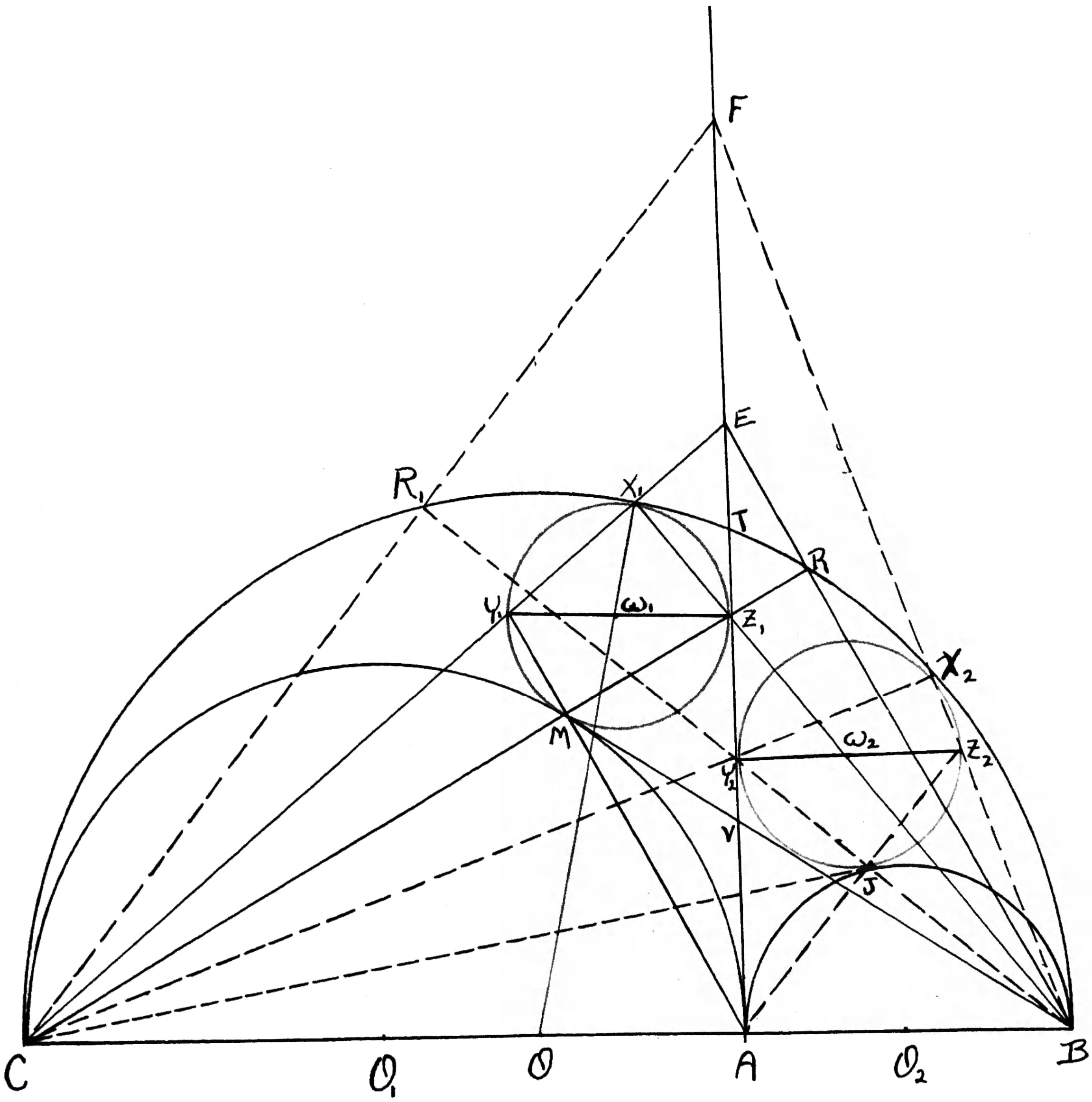


Figure 1

II. THEOREMS KNOWN TO ANCIENT GEOMETERS

A. LENGTH OF THE PERIMETER. (Figure 1)

In the arbelos CTBJAM, that is the curvilinear figure contained by the three semi-circumferences CTB, BJA, and AMC, the perimeter, p, is equal to the circumference of the circle on CB.

Proof: $\overline{AC} + \overline{AB} = \overline{CB}$

$$p = \frac{\pi \overline{AC}}{2} + \frac{\pi \overline{AB}}{2} + \frac{\pi \overline{CB}}{2}$$

$$= \frac{\pi}{2}(\overline{AC} + \overline{AB} + \overline{CB})$$

$$= \frac{\pi}{2}(2\overline{CB})$$

$$= \pi \overline{CB}$$

B. AREA. (Figure 1)

The area of the arbelos CTBJAM is equal to that of the circle whose diameter is AT, the common tangent at A to BJA and AMC.

Proof: Let a = radius of circle CTB,

b = radius of circle AMC, and

c = radius of circle AJB.

$$\text{Area} = \frac{\pi a^2}{2} - \frac{\pi b^2}{2} - \frac{\pi c^2}{2}$$

$$= \frac{\pi}{2}(a^2 - b^2 - c^2)$$

$$= \frac{\pi}{2}\left(\left[\frac{CB}{2}\right]^2 - \left[\frac{CA}{2}\right]^2 - \left[\frac{AB}{2}\right]^2\right)$$

$$= \frac{\pi}{8}\left(\overline{CB}^2 - \overline{CA}^2 - \overline{AB}^2\right)$$

$$\text{but } \overline{CB}^2 = (CA + AB)^2$$
$$= \overline{CA}^2 + \overline{AB}^2 + 2\,\overline{CA}\cdot\overline{AB}$$
$$\text{but } \overline{AT}^2 = \overline{CA}\cdot\overline{AB}$$
$$= \overline{CA}^2 + \overline{AB}^2 + 2\,\overline{AT}^2$$
$$\therefore \text{Area} = \frac{\pi}{8}\left(\overline{CA}^2 + \overline{AB}^2 + 2\,\overline{AT}^2 - \overline{CA}^2 - \overline{AB}^2\right)$$
$$= \frac{\pi}{8}\left(2\,\overline{AT}^2\right)$$
$$= \pi\left(\frac{AT}{2}\right)^2$$

C. <u>INSCRIBED CIRCLES</u>. (Figure 1)

1. <u>The two circles inscribed in the arbelos and tangent to **AT** are equal</u>.*

Proof: Let Z_1X_1M and X_2Y_2J be the two circles. Draw diameters Z_1Y_1 and Y_2Z_2 parallel to CB. X_1 is the external homothetic center of circle CB and circle Z_1X_1M; for $O\omega_1$, the line of centers, goes through the point of tangency X_1. Since $Y_1\omega_1$ and CO_2 are parallel radii, $\frac{X_1\omega_1}{X_1O} = \frac{Y_1\omega_1}{CO}$. Therefore X_1 divides the line of centers into the ratio of the radii and is therefore the external homothetic center. Then since CY_1 joins the ends of parallel radii, it passes through the homothetic center X_1. Similarly, B, Z_1 and X_1 are collinear.

M, the point of tangency of circle CA and circle Z_1X_1M, lies on the line of centers and is the internal homothetic center since it divides the line of centers into

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* Archimedes, Lemma 5.

the ratio of the radii. The points C and Z_1 are the opposite ends of parallel diameters; therefore, C, M and Z_1 are collinear. Similarly Y_1, M and A are collinear.

Let CX_1 meet AT produced at E, and let CZ_1 meet circle CB at R. Draw BR.

Consider triangle ECB:

$EA \perp CB$ by construction, and

$BX_1 \perp CE$ since angles inscribed in a semi-circle are right angles. Therefore Z_1 is the orthocenter of triangle ECB. Hence CZ_1 produced will be perpendicular to BR. But CR is perpendicular to BR since they form an angle inscribed in a semi-circle. Hence B, R, E are on a line. Since angle CMA and angle CRB are inscribed in a semi-circle, they are right angles and MA is parallel to RB.

Hence $\dfrac{CB}{BA} = \dfrac{CE}{Y_1E}$

From $\triangle CAE$ and $\triangle Y_1Z_1E$

$$\frac{CE}{Y_1E} = \frac{CA}{Y_1Z_1}$$

Then $\dfrac{CB}{BA} = \dfrac{CA}{Y_1Z_1}$

From this equation $Y_1Z_1 = \dfrac{BA \cdot CA}{CB}$

Consider the circle ω_2.

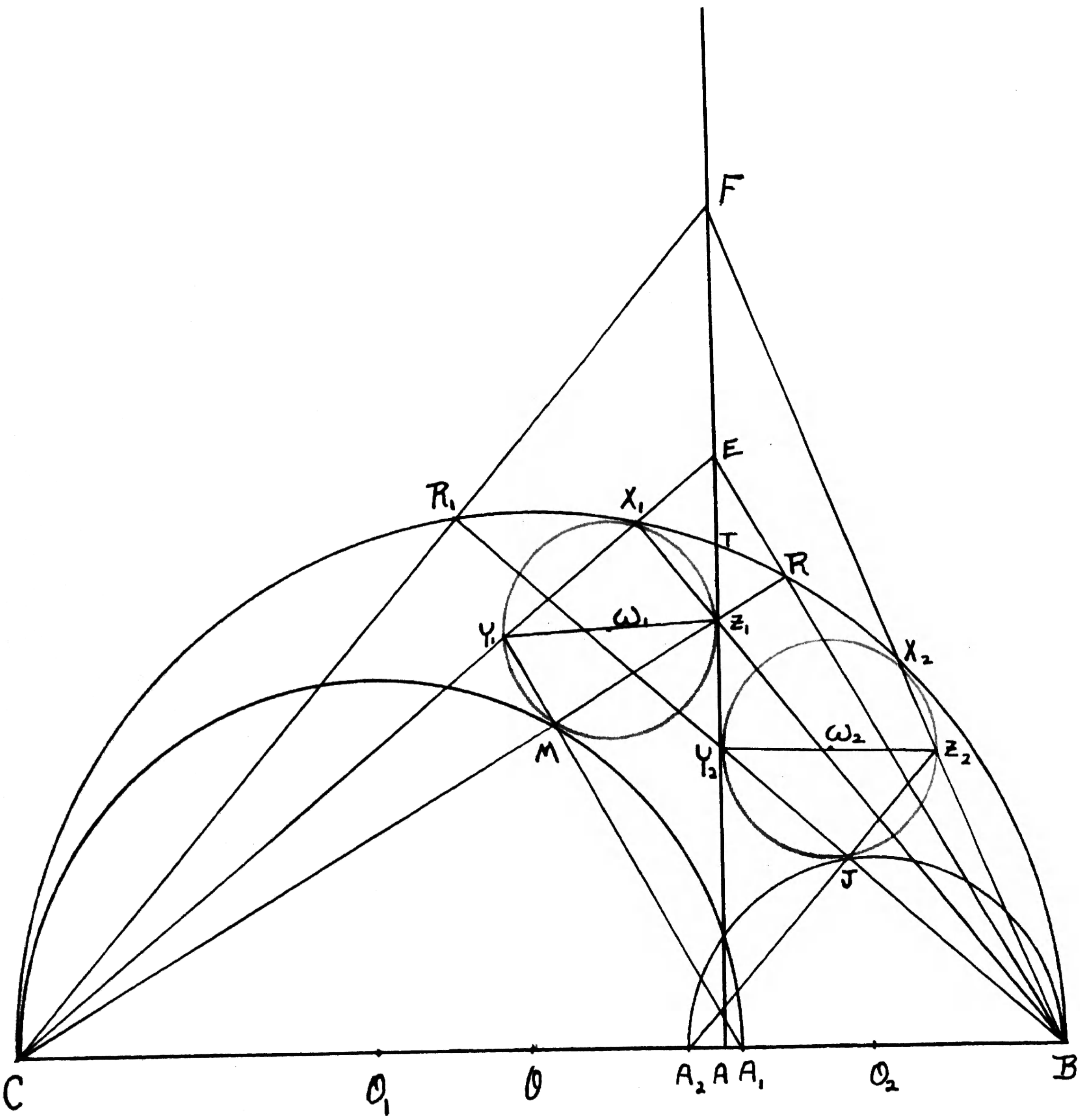

Figure 2

The point X_2 is the external homothetic center of the circles ω_2 and CTB, and J is the internal homothetic center of the circles ω_2 and ATB. Therefore B, Z_2, X_2 and B, J, Y_2 are collinear.

Extend BX_2 to F.

$\angle CR_1B$ is a right angle.

$FA \perp CB$

$CX_2 \perp FB$

$\therefore$ Y_2 = orthocenter of $\triangle CFB$.

$\therefore$ Y_2B is the altitude from B and CFR_1 is a straight line.

$\angle CR_1B$ and $\angle AJB$ are right angles.

$\therefore$ $AZ_2 \parallel CF$.

$$\frac{CB}{CA} = \frac{FB}{FZ_2} = \frac{AB}{Y_2Z_2}$$

$$Y_2Z_2 = \frac{CA \cdot AB}{CB}$$

$\therefore$ $Y_1Z_1 = Y_2Z_2$.

2. Equality of inscribed circles when the small semi-circles intersect. (Figure 2)

<u>If the circles CMA and BJA intersect, the circles Z_1X_1M and X_2Y_2J are equal, provided AT is the radical axis of circle CMA_1 and circle BJA_2.</u>

Proof: (Use the same reasoning as in the last proof.)

The point X_1 is the external homothetic center of circles CX_1B and $X_1Y_1Z_1$. Therefore C, Y_1, X_1 and B, Z_1, X_1 are collinear for they are on the lines joining the ends of parallel radii. The point M is the internal homothetic center of circles CMA_1 and $X_1Y_1Z_1$. Therefore C, M, Z_1 and A_1, M, Y_1 are collinear for, again, they are on the lines joining the ends of parallel radii. Let CX_1 produced meet AT produced at E, and CZ_1 meet circle CTB at R. Join B and R.

Consider triangle CEB.

$EA \perp CB$ hence is the altitude from E, and

$BX_1 \perp CE$ hence is the altitude from B.

They meet in Z_1 which is therefore the orthocenter of triangle CEB. Hence the altitude from C goes through Z_1. But CR is perpendicular to BR since they form an angle inscribed in a semi-circle, so R lies on BE.

$$\angle CMA_1 = \angle CRB = 90^\circ \qquad \therefore MA_1 \parallel RB.$$

Hence $\dfrac{CB}{A_1B} = \dfrac{CE}{Y_1E}$.

$$\triangle CEA \sim \triangle EY_1Z_1$$

$$\frac{CE}{Y_1E} = \frac{CA}{Y_1Z_1}$$

Hence $\dfrac{CB}{A_1B} = \dfrac{CA}{Y_1Z_1}$,

or
$$Y_1Z_1 = \frac{CA \cdot A_1B}{CB}$$
$$= \frac{CA(BA - AA_1)}{CB}$$
$$= \frac{CA \cdot BA}{CB} - \frac{CA \cdot AA_1}{CB}.$$

Consider the circle ω_2.

The point J is the internal homothetic center of circles A_2JB and $X_2Y_2Z_2$. Therefore Y_2, J, B and Z_2, J, A are on a line. X_2 is the external homothetic center of circles CTB and $X_2Y_2Z_2$. Therefore C, Y_2, X_2 and X_2, Z_2, B are collinear. Extend X_2Z_2B to meet AT in F and BY_2 to meet circle CTB in R_1. Join C and R_1.

Consider triangle CFB.

FA is the altitude from F, and CX_2 is the altitude from C. Therefore Y_2 is the orthocenter of triangle CFB. Hence the altitude from B goes through Y_2. But BR_1 is perpendicular to CR_1, so R_1 lies on CF.

$$\angle BJA_2 = \angle BR_1C = 90^\circ$$

$$\therefore\ A_2JZ_2 \parallel CR_1F$$

Hence $\dfrac{Z_2F}{BF} = \dfrac{CA_2}{CB}$

But $\triangle BFA \sim \triangle FY_2Z_2$

$$\therefore\ \frac{Z_2F}{BF} = \frac{Y_2Z_2}{AB}$$

Hence $\dfrac{Y_2Z_2}{AB} = \dfrac{CA_2}{CB}$ or $Y_2Z_2 = \dfrac{CA_2 \cdot AB}{CB}$

$$Y_2Z_2 = \frac{(CA - A_2A)AB}{CB} = \frac{AB \cdot CA}{CB} - \frac{AB \cdot A_2A}{CB}$$

Since AF is the radical axis of circles CM_1A_1 and A_2JB, $CA \cdot AA_1 = BA \cdot AA_2$.

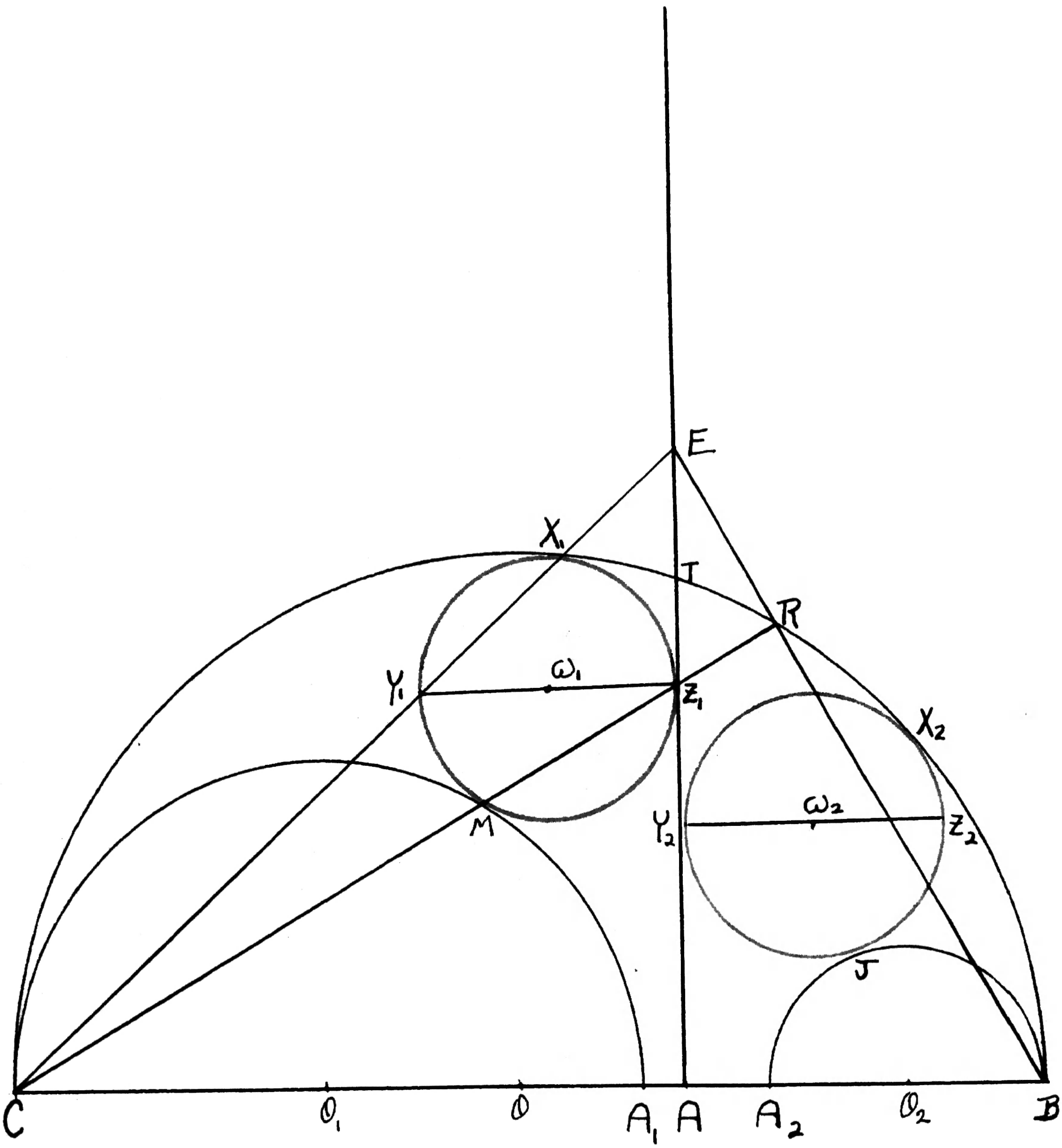

Figure 3

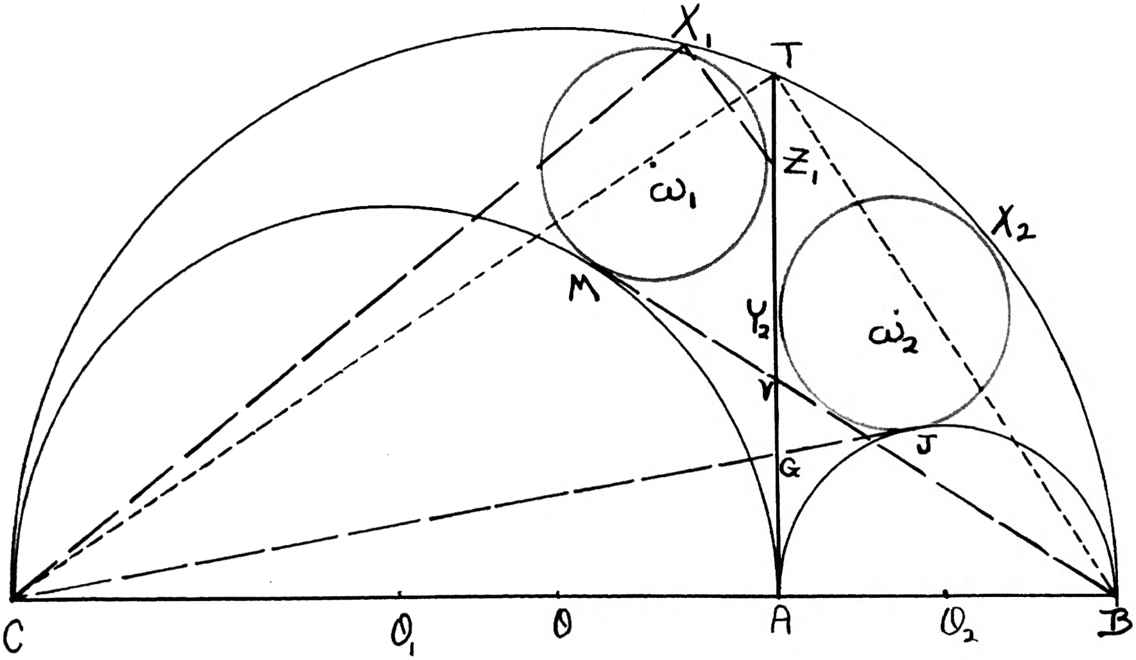

Figure 4

Hence by substitution $Y_1 Z_1 = \frac{CA \cdot BA}{CB} - \frac{BA \cdot AA_2}{CB}$

and $\therefore \; Y_1 Z_1 = Y_2' Z_2$.

3. Equality of inscribed circles when the small semicircles do not touch each other. (Figure 3)

<u>If the circles CMA and BJA have no point in common, the circles Z_1X_1M and X_2Y_2J are equal, provided AT is the radical axis of circle CMA_1 and circle BJA_2.</u>

The argument is exactly the same as that in article C 2. of this section, except $A_1B = AB + AA_1$ and $CA_2 = CA + AA_2$.

This extension of the theorem of article C, due to an Arabian mathematician, Alkauhi, is given in Borelli's "Apollonii Pergaei Conicorum." Lib. V, VI, VII and Archimedes Assumptorum Liber (Florentiae, 1661), pp. 393-5.

D. <u>COMMON TANGENTS</u>. (Figure 4)

1. Position.

<u>The common tangent to the two circles CMA and X_1MZ_1 at M passes through B, and the common tangent to the two circles AJB and X_2Y_2J at J passes through C.</u>

Proof: $\angle CAZ_1 = \angle CX_1Z_1 = 90^\circ$.

Therefore points C, A, Z_1, X_1 are concyclic; and $BC \cdot BA$ is equal to $BX_1 \cdot BZ_1$, that is, B has equal powers with respect to the two circles CMA and X_1MZ_1 and is therefore on the radical axis. M is also on the radical axis. Therefore BM is the radical axis or common tangent at M. Similarly for CJ.

2. Length.

$\underline{BM = BT}$ and $\underline{CJ = CT}$.

Proof:

$\overline{BM}^2 = \overline{BC} \cdot \overline{BA}$ for the tangent is the mean proportional between the whole secant and the external segment.

$\overline{AT}^2 = \overline{CA} \cdot \overline{AB}$ since "the half chord drawn perpendicular to the diameter of a circle is the mean proportional between the segments into which it divides the diameter."

In triangle ATB

$\overline{AT}^2 + \overline{AB}^2 = \overline{TB}^2$

$\overline{CA} \cdot \overline{AB} + \overline{AB}^2 = \overline{TB}^2$

$\overline{AB} \cdot \overline{CB} = \overline{TB}^2$

$\therefore \overline{BM}^2 = \overline{TB}^2$ or $\overline{BM} = \overline{TB}$.

Similarly for $\overline{CJ} = \overline{CT}$.

3. Common tangents and radical axis.

<u>The line BM bisects AZ_1 at V, and CJ bisects AY_2 at G.</u>

Proof:

The radical axis of two circles bisects their common tangent. The line BM is the radical axis of circles CMA and X_1Z_1M, and AT is their common tangent; the line CJ is the radical axis of circles AJB and $X_2Y_2Z_2$ and AT is their common tangent.

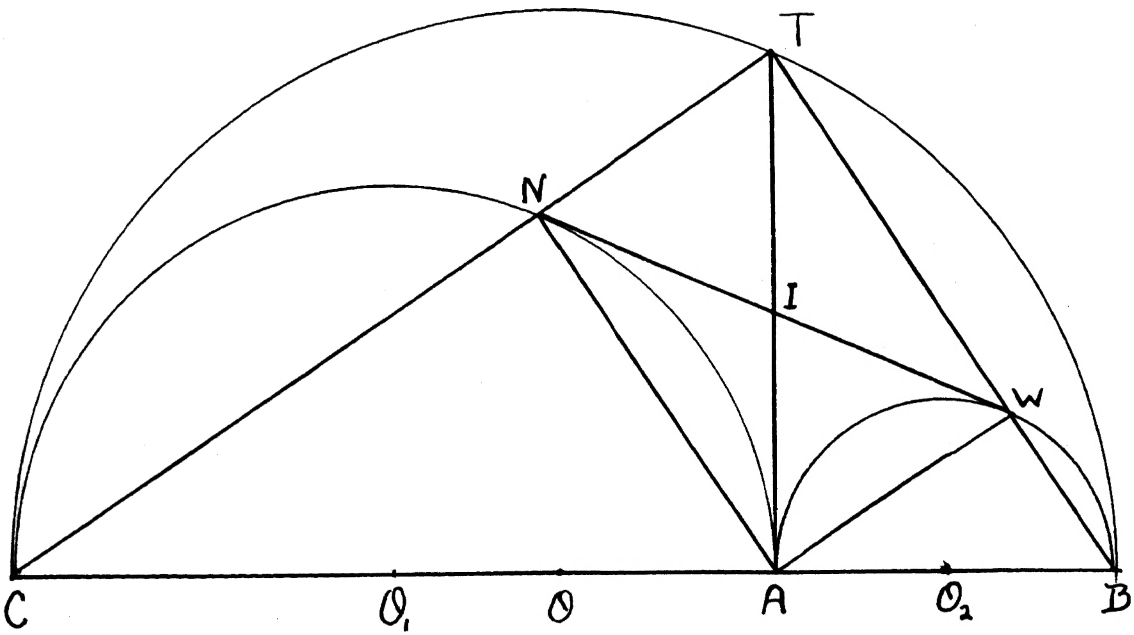

Figure 5

4. Construction of inscribed circles.

This leads to a method of finding ω_1 and ω_2, the centers of the circles X_1MZ_1 and Y_2JX_2.

From B draw BM tangent to the circle CMA, and cutting AT in V; make VZ_1 equal to VA; through Z_1 draw Z_1Y_1 parallel to CB. If O_1 is the center of the circle CMA, O_1M will meet Z_1Y_1 in ω_1.

From C draw CJ tangent to the circle AJB, and cutting AT in G. Make AG equal to GY_2. Through Y_2 draw Y_2Z_2 parallel to CB. If O_2 is the center of circle AJB, draw O_2J to meet Y_2Z_2 in ω_2.

E. COMMON TANGENT TO THE SEMI-CIRCLES AB AND AC. (Figure 5)

<u>If CT cuts the circle on CA at N, and BT cuts the circle on AB at W, NW is a common tangent to circle CNA and circle AWB</u>.

Proof:

Join A, N and A, W and let AT and NW intersect in I. Since angles CNA, CTB, and AWB are each inscribed in a semi-circle, they are right angles and AWTN is a rectangle. The lines IN and IW are each equal to IA for they are halfs of diagonals of a rectangle. Therefore IN and IW are tangents to circle CNA and circle AWB respectively, since AT is the radical axis of these two circles.

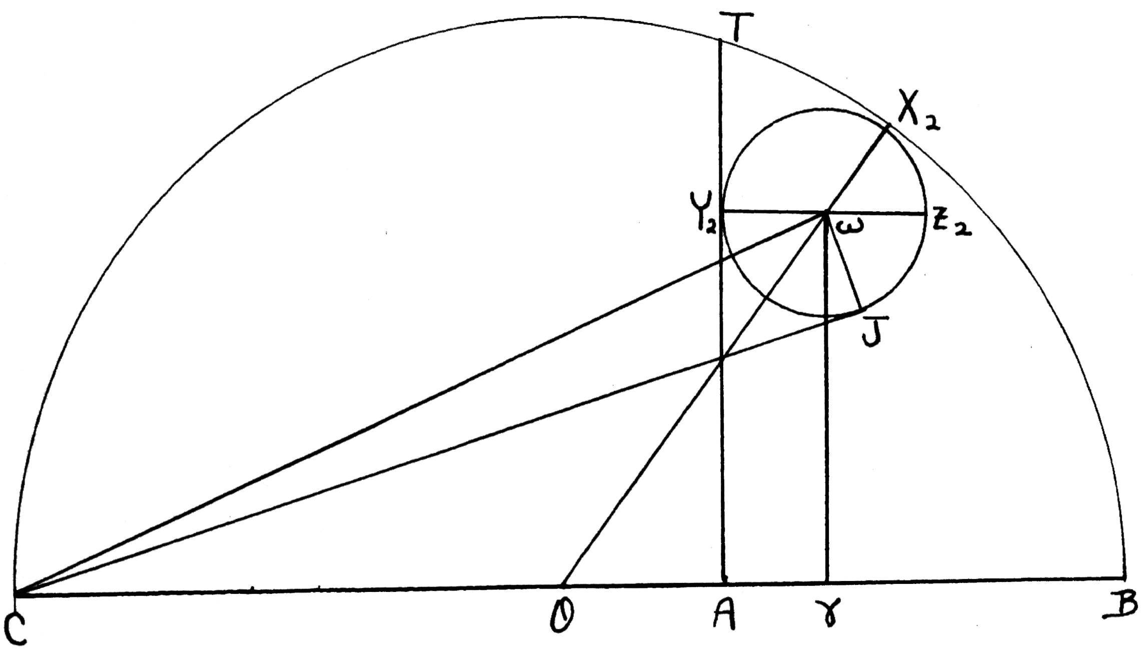

Figure 6

F. EXTENSION OF THE THEOREM IN ARTICLE D OF THIS SECTION. (Figure 6)

The theorem of article D (1) is a particular case of the following theorems:*

1. Circles tangent internally.

Let CTB be a semi-circle, and AT be perpendicular to CB. If a variable circle $X_2Y_2Z_2$ is drawn tangent to AT and the arc BT, and from C a tangent CJ be drawn to it, the length of CJ is constant.

Proof:

Let O and ω be the centers of circle CTB and circle X_2Y_2J respectively. $O\omega$ will pass through X_2, the point of tangency. Join ω and C, ω and J and draw $\omega\gamma$ perpendicular to CB.

Then $\overline{C\omega}^2 = \overline{C\gamma}^2 + \overline{\gamma\omega}^2$

$$= (\overline{CO} + \overline{O\gamma})^2 + \overline{\gamma\omega}^2$$

$$= \overline{CO}^2 + 2\overline{CO}\cdot\overline{O\gamma} + \overline{O\gamma}^2 + \overline{\gamma\omega}^2$$

but $\overline{O\gamma}^2 + \overline{\gamma\omega}^2 = \overline{O\omega}^2$

$$= \overline{CO}^2 + 2\overline{CO}\cdot\overline{O\gamma} + \overline{O\omega}^2$$

$$= \overline{CO}^2 + \overline{O\omega}^2 + 2\overline{CO}\cdot\overline{O\gamma}$$

$$= \overline{CO}^2 + (\overline{OX_2} - \overline{\omega X_2})^2 + 2\overline{CO}\cdot\overline{O\gamma}$$

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* Leybourn's Mathematical Repository, New Series, Vol. VI, Part I, pp. 209-11.

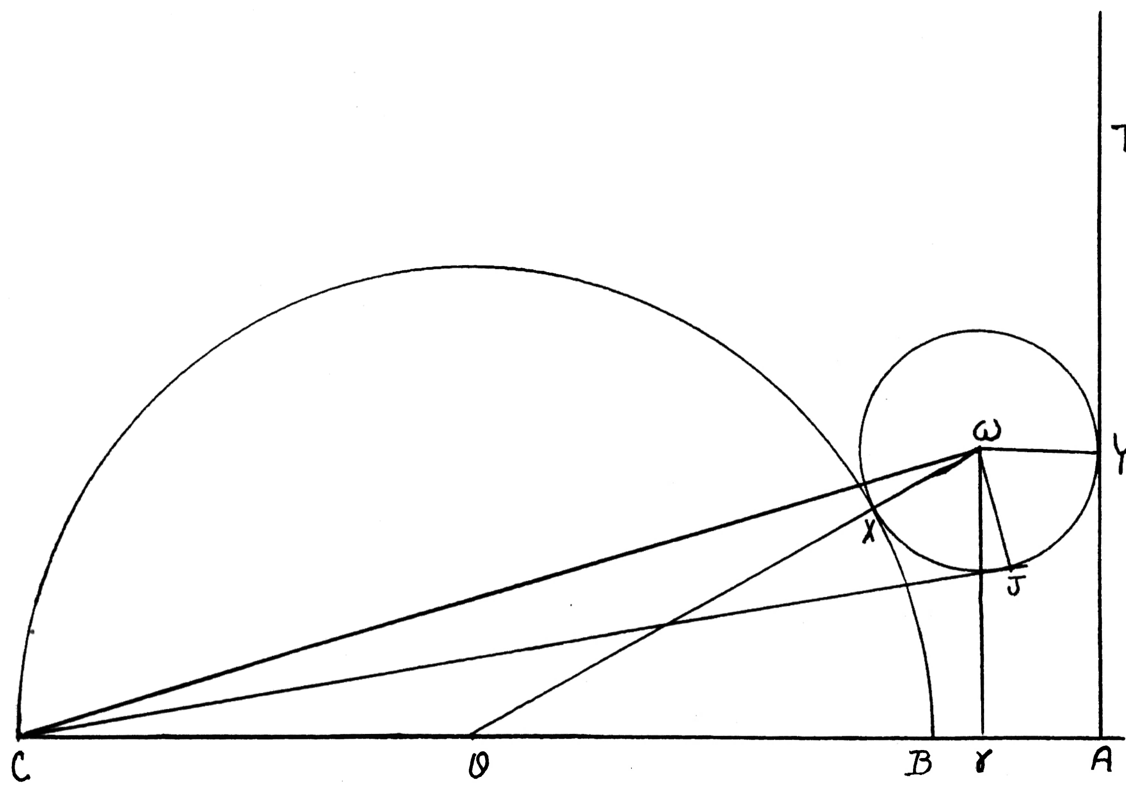

Figure 7

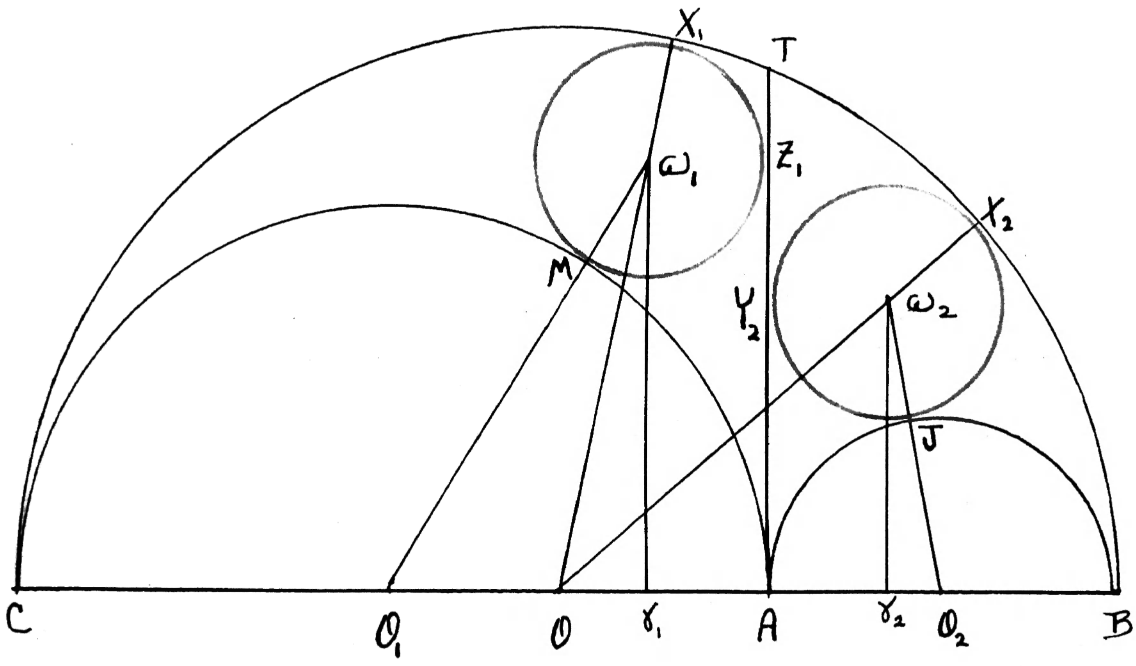

Figure 8

$$= \overline{CO}^2 + (\overline{CO} - \overline{A\gamma})^2 + 2\,\overline{CO}\cdot\overline{O\gamma}$$

$$= \overline{CO}^2 + \overline{CO}^2 - 2\,\overline{CO}\cdot\overline{A\gamma} + \overline{A\gamma}^2 + 2\,\overline{CO}\,(\overline{OA} + \overline{A\gamma})$$

$$= 2\,\overline{CO}^2 + 2\,\overline{CO}\cdot\overline{OA} + \overline{A\gamma}^2$$

$$= 2\,\overline{CO}\cdot\overline{CA} + \overline{A\gamma}^2$$

$$\therefore\ \overline{C\omega}^2 - \overline{A\gamma}^2 = 2\,\overline{CO}\cdot\overline{CA}$$

$$\text{or}\ \ \overline{C\omega}^2 - \overline{\omega J}^2 = 2\,\overline{CO}\cdot\overline{CA}$$

$$\text{Hence}\ \ \overline{CJ}^2 = 2\,\overline{CO}\cdot\overline{CA}.$$

2. Circles tangent externally. (Figure 7)

The theorem is still true if the variable circle touches AT produced and the arc CB externally. It is also true when AT, the perpendicular to CB, touches the semicircle, or falls entirely outside, the contact in these cases being necessarily external.

G. A SECOND PROOF OF EQUALITY OF INSCRIBED CIRCLES. (Figure 8)

Let O, O_1, and O_2 be the centers of the semicircles CTB, CMA, and AJB, and let ω_1, ω_2 be the centers of the circles X_1MZ_1 and X_2Y_2J. From ω_1 and ω_2 draw $\omega_1\gamma_1$ and $\omega_2\gamma_2$ perpendicular to CB; and join $O\omega_1$, $O_1\omega_1$, $O\omega_2$, and $O_2\omega_2$. Then $O\omega_2$ passes through X_2, $O_2\omega_2$ through J, $O\omega_1$ through X_1, and $O_1\omega_1$ through M.*

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* The Gentleman's Diary, 1833, p. 40.

Proof: $A\gamma_2$ = radius of circle X_2Y_2J and

$A\gamma_1$ = radius of circle X_1MZ_1

$\underline{CO} = \frac{1}{2}CB = \frac{1}{2}(CA+AB) = \frac{1}{2}CA + \frac{1}{2}AB = \underline{O_1O_2}$

$O_1O = CO - CO_1 = O_1O_2 - O_1A = AO_2$

$CO_1 = O_1O + OA = AO_2 + OA = OO_2$

Now $\overline{O_1\omega_1}^2 - \overline{O\omega_1}^2 = \overline{O_1\gamma_1}^2 - \overline{O\gamma_1}^2$

$$\text{for } \overline{O_1\omega_1}^2 = \overline{\gamma_1\omega_1}^2 + \overline{O_1\gamma_1}^2$$

$$\overline{O\omega_1}^2 = \overline{O\gamma_1}^2 + \overline{\gamma_1\omega_1}^2.$$

But $\overline{O_1\omega_1}^2 - \overline{O\omega_1}^2 = (\overline{O_1M} + \overline{M\omega_1})^2 - (\overline{OX_1} - \overline{X_1\omega_1})^2$

$$= (\overline{O_1A} + \overline{A\gamma_1})^2 - (\overline{CO} - \overline{A\gamma_1})^2$$

$$= \overline{O_1A}^2 - \overline{CO}^2 + 2\,\overline{O_1A}\cdot\overline{A\gamma_1} + 2\,\overline{CO}\cdot\overline{A\gamma_1}$$

$$= \overline{O_1A}^2 - \overline{CO}^2 + 2\,\overline{A\gamma_1}(\overline{O_1A} + \overline{CO})$$

$$\text{but } (\overline{O_1A} + \overline{CO}) = (\overline{O_1A} + \overline{O_1O_2})$$

$$= (\overline{CO_1} + \overline{O_1O_2})$$

$$= \overline{O_1A}^2 - \overline{CO}^2 + 2\,\overline{A\gamma_1}\cdot\overline{CO_2}.$$

And $\overline{O_1\gamma_1}^2 - \overline{O\gamma_1}^2 = (\overline{O_1A} - \overline{A\gamma_1})^2 - (\overline{OA} - \overline{A\gamma_1})^2$

$$= \overline{O_1A}^2 - \overline{OA}^2 - 2\,\overline{O_1A}\cdot\overline{A\gamma_1} + 2\,\overline{OA}\cdot\overline{A\gamma_1}$$

$$= \overline{O_1A}^2 - \overline{OA}^2 - 2\,\overline{A\gamma_1}(\overline{O_1A} + \overline{AO})$$

$$= \overline{O_1A}^2 - \overline{OA}^2 - 2\,\overline{A\gamma_1}\cdot\overline{O_1O}.$$

Hence $\overline{O_1A}^2 - \overline{OA}^2 - 2\,\overline{O_1O}\cdot\overline{A\gamma_1} = \overline{O_1A}^2 - \overline{CO}^2 + 2\,\overline{CO_2}\cdot\overline{A\gamma_1}$

and $\therefore \quad \overline{CO}^2 - \overline{OA}^2 = 2\,\overline{A\gamma_1}(\overline{CO_2} + \overline{O_1O})$

$$= 2\,\overline{A\gamma_1} \cdot \overline{CB}$$

again $\overline{O\omega_2}^2 - \overline{O_2\omega_2}^2 = \overline{O\gamma_2}^2 - \overline{O_2\gamma_2}^2$

for $\quad \overline{O\omega_2}^2 = \overline{O\gamma_2}^2 + \overline{\gamma_2\omega_2}^2$

and $\quad \overline{O_2\omega_2}^2 = \overline{O_2\gamma_2}^2 + \overline{\gamma_2\omega_2}^2$

but $\overline{O\omega_2}^2 - \overline{O_2\omega_2}^2 = (\overline{OX_2} - \overline{X_2\omega_2})^2 - (\overline{O_2J} + \overline{J\omega})^2$

$$= (\overline{CO} - \overline{A\gamma_2})^2 - (\overline{AO_2} + \overline{A\gamma_2})^2$$

$$= \overline{CO}^2 - \overline{AO_2}^2 - 2\,\overline{CO} \cdot \overline{A\gamma_2} - 2\,\overline{AO_2} \cdot \overline{A\gamma_2}$$

$$= \overline{CO}^2 - \overline{AO_2}^2 - 2\,\overline{A\gamma_2}\,(\overline{CO} + \overline{AO_2})$$

$$= \overline{CO}^2 - \overline{AO_2}^2 - 2\,\overline{A\gamma_2}(\overline{CO} + \overline{O_1O})$$

$$= \overline{CO}^2 - \overline{AO_2}^2 - 2\,\overline{A\gamma_2} \cdot \overline{O_1B}\ ;$$

and $\overline{O\gamma_2}^2 - \overline{O_2\gamma_2}^2 = (\overline{OA} + \overline{A\gamma_2})^2 - (\overline{AO_2} - \overline{A\gamma_2})^2$

$$= \overline{OA}^2 - \overline{AO_2}^2 + 2\,\overline{OA} \cdot \overline{A\gamma_2} + 2\,\overline{AO_2} \cdot \overline{A\gamma_2}$$

$$= \overline{OA}^2 - \overline{AO_2}^2 + 2\,\overline{A\gamma_2}\,(\overline{OA} + AO_2)$$

$$= \overline{OA}^2 - \overline{AO_2}^2 + 2\,\overline{A\gamma_2} \cdot \overline{OO_2}$$

$$= \overline{OA}^2 - \overline{AO_2}^2 + 2\,\overline{A\gamma_2} \cdot \overline{CO_1}\ .$$

Therefore $\overline{CO}^2 - \overline{AO_2}^2 - 2\,\overline{A\gamma_2} \cdot \overline{O_1B} = \overline{OA}^2 - \overline{AO_2}^2 + 2\,\overline{A\gamma_2} \cdot \overline{CO_1}$.

Hence $\quad \overline{CO}^2 - \overline{OA}^2 = 2\,\overline{A\gamma_2}\,(\overline{CO_1} + \overline{O_1B})$

$$= 2\,\overline{A\gamma_2} \cdot \overline{CB}\ .$$

Hence $\cancel{2\overline{CB}} \cdot \overline{A\gamma_2} = \cancel{2\overline{CB}} \cdot \overline{A\gamma_1}$ and the two circles are therefore equal.

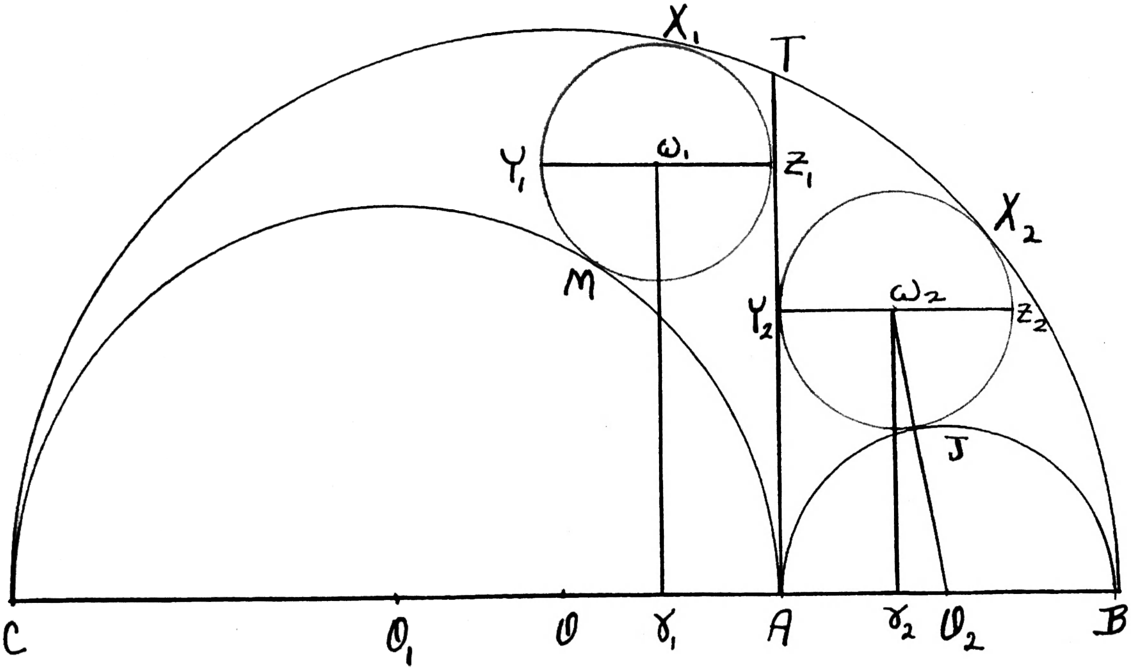

Figure 9

H. INVERSE POINTS AND LINE SEGMENT RELATIONSHIPS. (Figure 9)

1. Inverse points.

The points O_2 and γ_1 are inverse with respect to circle CMA, and the points O_1 and γ_2 are inverse with respect to circle AJB.

Proof: $O_1O_2 = \dfrac{CB}{2}$

$$O_1\gamma_1 = O_1A - A\gamma_1 \qquad \text{but } A\gamma_1 = \frac{(CO-OA)(CO+OA)}{2CB}$$

$$= \frac{CA}{2} - \frac{BA \cdot CA}{2CB} \qquad\qquad = \frac{(\frac{1}{2}CB - OA)\,AC}{2CB}$$

$$= \frac{CA \cdot CB - BA \cdot CA}{2CB} \qquad\qquad = \frac{BA \cdot AC}{2CB}$$

$$= \frac{CA\,(CB - BA)}{2CB}$$

$$= \frac{\overline{CA}^2}{2CB}$$

$$\text{Hence} \quad O_1O_2 \cdot O_1\gamma_2 = \frac{CB}{2} \cdot \frac{\overline{CA}^2}{2CB}$$

$$= \frac{\overline{CA}^2}{4}$$

$$= \overline{O_1A}^2$$

$\therefore$ O_2 and γ_1 are inverse points with respect to circle CMA. Similarly for O_1 and γ_2.

$$O_1O_2 = \frac{CB}{2}$$

$$\begin{aligned}
O_2\gamma_2 &= O_2A - A\gamma_2 \\
&= O_2A - \frac{BA \cdot CA}{2CB} \\
&= \frac{AB}{2} - \frac{AB \cdot CA}{2CB} \\
&= \frac{AB \cdot CB - AB \cdot CA}{2CB} \\
&= \frac{AB(CB - CA)}{2CB} \\
&= \frac{\overline{AB}^2}{2CB}
\end{aligned}$$

$$\begin{aligned}
\therefore\ O_1O_2 \cdot O_2\gamma_2 &= \frac{1}{2}CB \cdot \frac{\overline{AB}^2}{2CB} \\
&= \frac{\overline{AB}^2}{4} \\
&= \overline{O_2A}^2
\end{aligned}$$

$\therefore$ O_1 and γ_2 are inverse points with respect to circle AJB.

2. Relation between line of centers and radii of small semi-circles.

$\underline{O_2\omega_2 + O_2\gamma_2 = AB}$ <u>and</u> $\underline{O_1\omega_1 + O_1\gamma_1 = CA}$.

Proof: $O_2\omega_2 = O_2J + J\omega_2$

$= AO_2 + A\gamma_2$

$= AO_2 + A\gamma_1$

$= O_2\gamma_1$

$\therefore O_2\omega_2 + O_2\gamma_2 = O_2\gamma_1 + O_2\gamma_2$

$= O_2\gamma_2 + \gamma_2\gamma_1 + O_2\gamma_1$

$= 2O_2\gamma_2 + 2\gamma_2 A$

$= 2(O_2\gamma_2 + \gamma_2 A)$

$= 2O_2A$

$= AB$

$O_1\omega_1 = O_1M + M\omega_1$

$= O_1A + A\gamma_1$

$= O_1A + A\gamma_2.$

$= O_1\gamma_2$

$\therefore O_1\omega_1 + O_1\gamma_1 = O_1\gamma_2 + O_1\gamma_1$

$= O_1A + \gamma_1A + O_1\gamma_1$

$= O_1A + AO_1$

$= 2O_1A$

$= AC$

3. Another relation between the line of centers and the radii of small semi-circles.

$\underline{O\omega_2 + O\gamma_2 = CA}$ and $\underline{O\omega_1 - O\gamma_1 = AB}$.

Proof: $O\omega_2 = OX_2 - \omega_2X_2$

$O\gamma_2 = OA + A\gamma_2$

Hence $O\omega_2 + O\gamma_2 = OX_2 + OA$

$= OC + OA$

$= CA$

$O\omega_1 = OX_1 - \omega_1X_1$

$O\gamma_1 = OA - A\gamma_1$

Hence $O\omega_1 - O\gamma_1 = OX_1 - OA$

$= OB - OA$

$= AB$

I. RELATIONS BETWEEN $O_1\omega_1$ and $O_2\omega_2$. (Figure 9)

1. Their values.

$$O_1\omega_1 = O_1M + M\omega_1 \quad \text{and} \quad O_2\omega_2 = O_2J + J\omega_2$$

$$= \frac{CA}{2} + O_1\gamma_1 \qquad\qquad = \frac{AB}{2} + A\gamma_2$$

$$= \frac{CA}{2} + \frac{AB\cdot CA}{2CB} \qquad\qquad = \frac{AB}{2} + \frac{AB\cdot CA}{2CB}$$

$$= \frac{CA\cdot BC + AB\cdot CA}{2CB} \qquad\qquad = \frac{AB\cdot CB + AB\cdot CA}{2CB}$$

$$= \frac{CA(CB+AB)}{2CB} \qquad\qquad = \frac{AB(CB+CA)}{2CB}$$

2. Their sum.

$$O_1\omega_1 + O_2\omega_2 = \frac{CA(CB+AB) + AB(CB+CA)}{2CB}$$

$$= \frac{CA+AB}{2} + \frac{CA\cdot AB + AB\cdot CA}{2CB}$$

$$= \frac{CA+AB}{2} + \frac{2CA\cdot AB}{2CB}$$

$$= \frac{CB}{2} + 2A\gamma_1$$

$$= \tfrac{1}{2}CB + Y_2Z_2.$$

3. Their difference.

$$O_1\omega_1 - O_2\omega_2 = \frac{CA(CB+AB) - AB(CB+CA)}{2CB}$$

$$= \frac{(CA-AB)CB}{2CB}$$

$$= \tfrac{1}{2}(CA - AB)$$

$$= \tfrac{1}{2}(OC+OA) - \tfrac{1}{2}(OB-OA)$$

$$= \tfrac{1}{2}\,(OC - OB + 2\,OA)$$

$$= OA$$

4. Their product.

$$O_1\omega_1 \cdot O_2\omega_2 = O\gamma_2 \cdot O_1\gamma_1$$

$$= (O_1A + A\gamma_2)(AO_2 + A\gamma_1)$$

$$= O_1A \cdot AO_2 + \underbrace{O_1A \cdot A\gamma_1 + AO_2 \cdot A\gamma_2} + A\gamma_2 \cdot A\gamma_1$$

$$= \frac{CA \cdot AB}{4} + O_1O_2 \cdot A\gamma_2 + \overline{A\gamma_2}^2$$

$$= \frac{CA \cdot AB}{4} + \frac{CA + AB}{2} \cdot \frac{CA \cdot AB}{2CB} + \frac{\overline{Y_2Z_2}^2}{4}$$

$$= \frac{CA \cdot AB \cdot CB + \overline{CA}^2 \cdot AB + \overline{AB}^2 \cdot CA}{4CB} + \frac{\overline{Y_2Z_2}^2}{4}$$

$$= \frac{CA \cdot AB\,(CB + CA + AB)}{4CB} + \frac{\overline{Y_2Z_2}^2}{4}$$

$$= \frac{CA \cdot AB\,(2CB)}{4CB} + \frac{\overline{Y_2Z_2}^2}{4}$$

$$= \tfrac{1}{4}\left(2\overline{AT}^2 + \overline{Y_2Z_2}^2\right) \qquad \frac{CA}{AT} = \frac{AT}{AB}$$

5. Their quotient.

$$\frac{O_1\omega_1}{O_2\omega_2} = \frac{CA\,(CB + AB)}{2\,CB} \cdot \frac{2\,CB}{AB\,(CB + CA)}$$

$$= \frac{CA\,(CB + AB)}{AB\,(CB + CA)}$$

$$= \frac{2CO_1\,(2BO + 2O_1O)}{2O_2B\,(2CO + 2OO_2)}$$

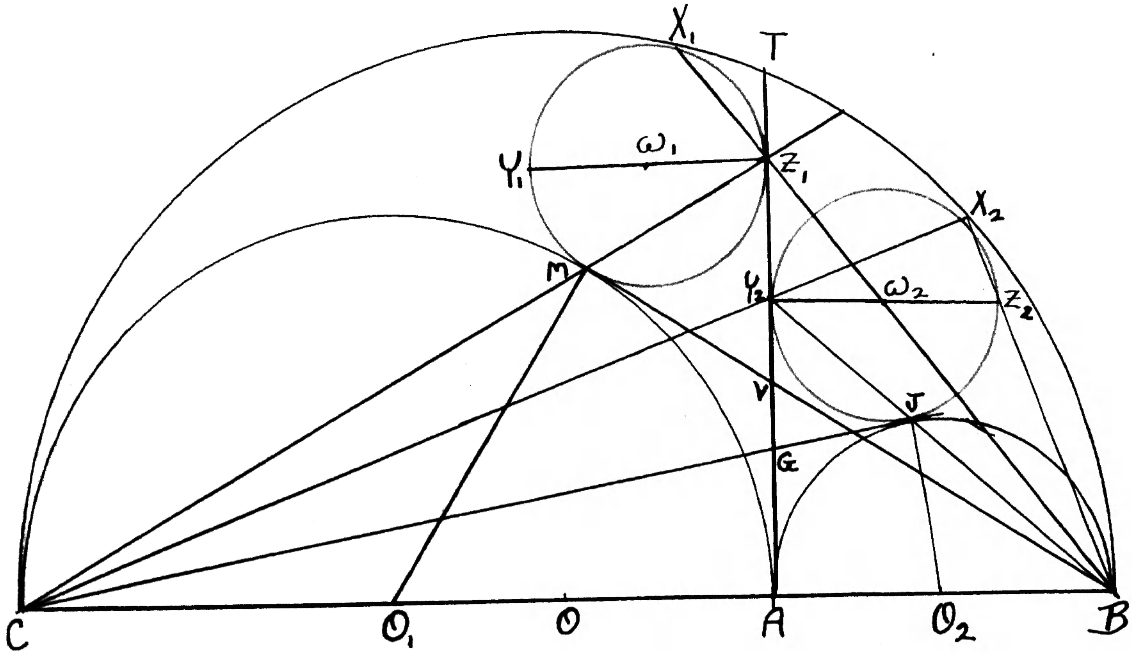

Figure 10

$$= \frac{2CO_1 \cdot 2BO_1}{2O_2B \cdot 2CO_2}$$

$$= \frac{CO_1 \cdot BO_1}{CO_2 \cdot BO_2}.$$

J. RELATIONS BETWEEN AZ_1 AND AY_2. (Figure 10)

1. Their values.

Rt. $\triangle BAV \sim$ rt. $\triangle BMO_1$ since they have $\angle B$ in common.

Hence $\dfrac{BA}{2AV} = \dfrac{BM}{2O_1M}$

$$\therefore \frac{BA}{AZ_1} = \frac{BT}{CA}$$

$$CL = \frac{CA \cdot BA}{BT}$$

$$= \frac{\overline{AT}^2}{BT}.$$

Similarly: rt. $\triangle CAG \sim$ rt. $\triangle CO_2J$.

$$\therefore \frac{CA}{2AG} = \frac{CJ}{2O_2J}$$

$$\frac{CA}{AY_2} = \frac{CT}{AB}$$

$$AY_2 = \frac{AB \cdot AC}{CT}$$

$$= \frac{\overline{AT}^2}{CT}.$$

2. Their sum and difference.

$$\text{Since } \triangle CBT \sim \triangle BAT, \quad \frac{CT}{CB} = \frac{AT}{BT}$$

$$AZ_1 \pm AY_2 = \overline{AT}^2\left(\frac{1}{BT} \pm \frac{1}{CT}\right)$$

$$= \overline{AT}^2\left(\frac{CT \pm BT}{BT \cdot CT}\right)$$

$$= \overline{AT}^2\left(\frac{CT \pm BT}{CB \cdot AT}\right)$$

$$= \frac{AT}{CB}\left(CT \pm BT\right).$$

3. The sum of their squares.

From the theorem, "If a straight line be a common tangent to two circles which touch each other externally, that part of the tangent between the points of contact is a mean proportional between the diameters of the circles," there results:

$$\overline{AZ_1}^2 = CA \cdot Z_1Y_1 \qquad \text{and} \qquad \overline{AY_2}^2 = AB \cdot Y_2Z_2$$

$$\text{but } Z_1Y_1 = Z_2Y_2$$

$$\therefore \overline{AZ_1}^2 + \overline{AY_2}^2 = (CA + AB)Y_2Z_2$$

$$= CB \cdot Y_2Z_2$$

$$= CA \cdot AB \quad (\text{from article C of this section})$$

$$= \overline{AT}^2.$$

Corollary.

$$\frac{CB}{AB} = \frac{\overline{AZ_1}^2}{\overline{Y_1Z_1}^2} \qquad \text{and} \qquad \frac{CB}{CA} = \frac{\overline{AY_2}^2}{\overline{Z_2Y_2}^2}.$$

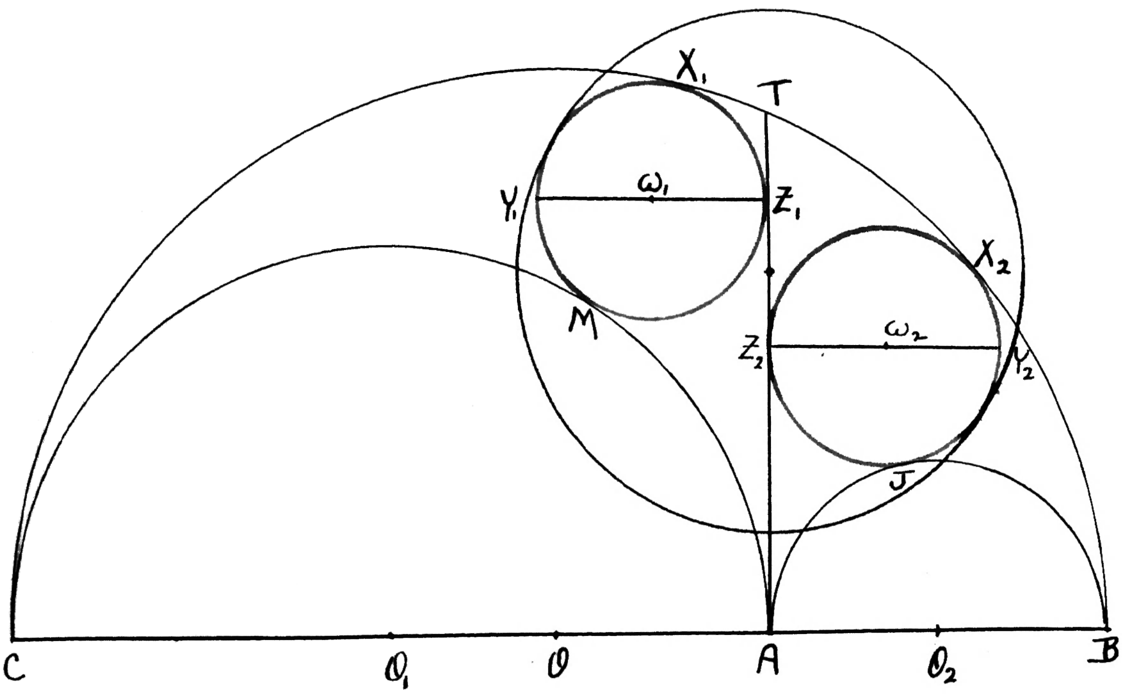

Figure 11

Proof:

$$\frac{CB}{AB} = \frac{CA}{Z_1Y_1} \quad \text{(from article C)} \qquad \frac{CB}{CA} = \frac{AB}{Y_2Z_2}$$

$$= \frac{CA \cdot Z_1Y_1}{\overline{Z_1Y_1}^2} \qquad\qquad = \frac{AB \cdot Y_2Z_2}{\overline{Y_2Z_2}^2}$$

$$= \frac{\overline{AZ_1}^2}{\overline{Z_1Y_1}^2} \qquad\qquad = \frac{\overline{AY_2}^2}{\overline{Y_2Z_2}^2}$$

4. Their product.

Since $\overline{AZ_1}^2 = CA \cdot Z_1Y_1$ and $\overline{AY_2}^2 = AB \cdot Y_2Z_2$

$$\overline{AZ_1}^2 \cdot \overline{AY_2}^2 = CA \cdot AB \cdot \overline{Y_2Z_2}^2$$

$$= \overline{AT}^2 \cdot \overline{Y_2Z_2}^2$$

Hence $AZ_1 \cdot AY_2 = AT \cdot Y_2Z_2$

5. Their quotient.

Since $AZ_1 \cdot BT = \overline{AT}^2$ (part (1) of this article)

and $AY_2 \cdot CT = \overline{AT}^2$

$$\frac{AZ_1}{AY_2} \cdot \frac{BT}{CT} = 1$$

or $$\frac{AZ_1}{AY_2} = \frac{CT}{BT}.$$

K. THE LEAST CIRCLE CIRCUMSCRIBING ω_1 AND ω_2. (Figure 11)

<u>The arbelos is equal to the least circle which can be circumscribed about the circles $X_1Y_1Z_1$ and $X_2Y_2Z_2$.</u>*

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* Leybourn's Mathematical Repository, New Series, Vol. VI, pp. 155, 214.

The diameter of the least circle that can be circumscribed to touch circle $X_1Y_1Z_1$ and circle $X_2Y_2Z_2$ will pass through ω_1 and ω_2 and be equal to $\omega_1\omega_2$ plus Z_2Y_2.

Proof:

$$\overline{\omega_1\omega_2}^2 = \overline{Y_2Z_1}^2 \qquad (\omega_1 Z_1 \parallel \omega_2 Y_2)$$

$$= \overline{Z_1Z_2}^2 + \overline{Z_2Y_2}^2$$

$$= \left(\frac{\overline{AT}^2}{BT} - \frac{\overline{AT}^2}{CT}\right) + \overline{Z_2Y_2}^2 \qquad [\text{article } J(1)]$$

$$= \frac{\overline{AT}^2}{\overline{CB}^2}(CB - BT)^2 + \frac{\overline{AT}^4}{\overline{CB}^2} \qquad [\text{article } J(2 \text{ and } 3)]$$

$$= \frac{\overline{AT}^2}{\overline{CB}^2}\left(\overline{CT}^2 - 2\, CT \cdot BT + \overline{BT}^2 + \overline{AT}^2\right)$$

$$\text{but } \overline{CT}^2 + \overline{BT}^2 = \overline{CB}^2$$

$$= \frac{\overline{AT}^2}{\overline{CB}^2}\left(\overline{CB}^2 - 2\, CT \cdot BT + \overline{AT}^2\right)$$

$$= \frac{\overline{AT}^2}{\overline{CB}^2}(CB - AT)^2$$

$$\therefore \omega_1\omega_2 = \frac{AT}{CB}(CB - AT)$$

$$= AT - \frac{\overline{AT}^2}{CB}$$

$$= AT - Z_2Y_2$$

$$\therefore \omega_1\omega_2 + Z_2Y_2 = AT.$$

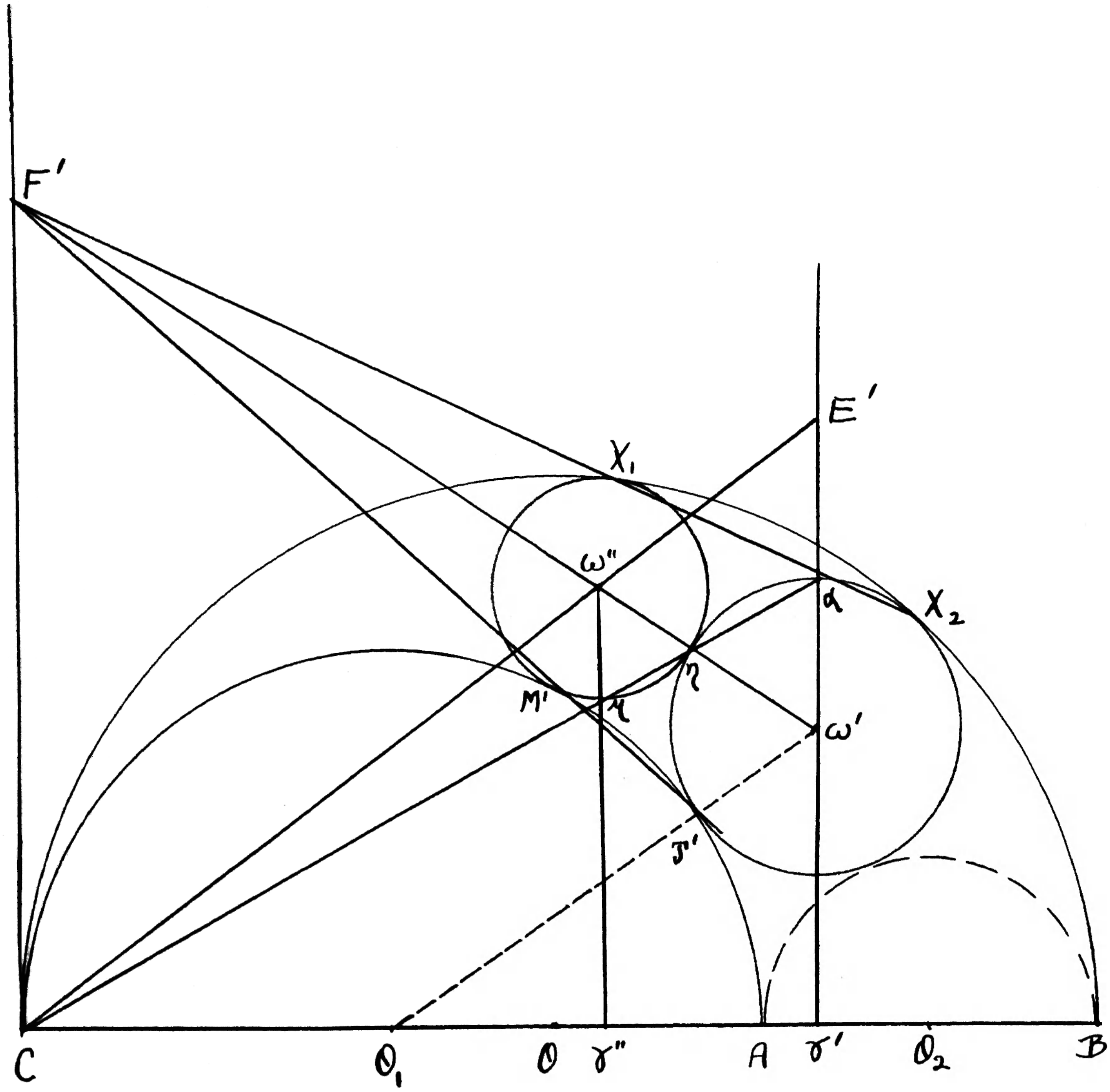

Figure 12

L. SUCCESSIVE INSCRIBED CIRCLES. (Figure 12)

If from ω' and ω'', the centers of two circles which are tangent to the semi-circles CX_2B and $CM'A$, and each other, there be drawn $\omega'\gamma'$ and $\omega''\gamma''$ ~~is~~ perpendicular to CB, and if r_1 and r_2 denote the radii of circles ω' and ω'' then

$$\frac{\omega'\gamma' + 2r_1}{2r_1} = \frac{\omega''\gamma''}{2r_2} \; *$$

Of the six homothetic centers of any three circles, every two internal centers are collinear with one external center and the three external centers are collinear. Since J' is the internal homothetic center of the circles ω' and $CM'A$, and M' the internal homothetic center of circles ω'' and $CM'A$, $J'M'$ produced passes through F', the external homothetic center of circles ω' and ω''. Since X_2 is the external homothetic center of circles ω' and CX_2B, and X_1 is the external homothetic center of ω'' and $CM'A$, X_2X_1 also passes through F'. Now with reference to F', the external homothetic center of circles ω' and ω'', the points J' and M' are anti-homothetic as are the points X_1 and X_2, and the two anti-homothetic points coincide at η.

$$\therefore \quad F'J' \cdot F'M' = F'\eta \qquad \text{and} \qquad F'X_2 \cdot F'X_1 = F'\eta.$$

Since F' has equal powers with respect to circles $CM'A$ and CX_2B, it is on the radical axis of those circles.

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* Pappus, Book IV, Proposition 15.

$\therefore\ \overline{F'C}^2 = F'J' \cdot F'M'$

$= \overline{F'\eta}^2$

and $F'C = F'\eta$.

Since $F'C = F'\eta$, and $\omega''\mu \parallel F'C$ and $\omega''\mu = \omega''\eta$,
then C, μ and η are on a line.

Since $\omega''\mu$ and $\omega'\alpha$ are parallel and opposite in direction, μ, η, and α are on a line.
Let $C\omega''$ meet $\omega'\alpha$ at E'.

Then $\dfrac{E'\alpha}{\omega''\mu} = \dfrac{C\alpha}{C\mu}$ from similar triangles,

$$= \frac{C\gamma'}{C\gamma''}$$

$$= \frac{\omega' F'}{\omega'' F'}$$

$$= \frac{r_1}{r_2} .$$

Then since $\omega''\mu = r_2$, $E'\alpha = r_1$ and $E'\omega' = 2r_1$,

$\dfrac{E'\gamma'}{\omega''\gamma''} = \dfrac{C\gamma'}{C\gamma''}$ from similar triangles and

$$\frac{E'\gamma'}{\omega''\gamma''} = \frac{r_1}{r_2} .$$

But $E'\gamma' = \omega'\gamma' + 2r_1$

$$\therefore\ \frac{\omega'\gamma' + 2r_1}{2r_1} = \frac{\omega''\gamma''}{2r_2} .$$

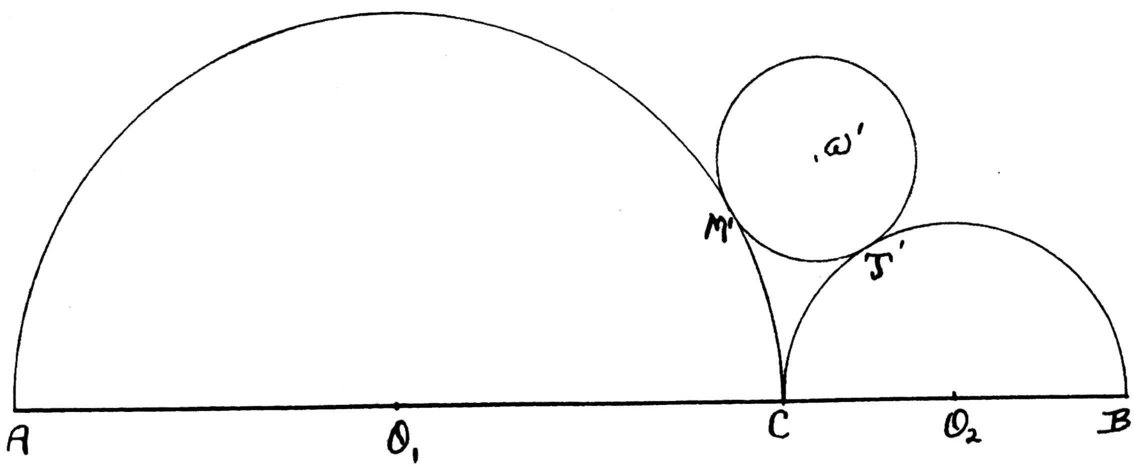

Figure 13

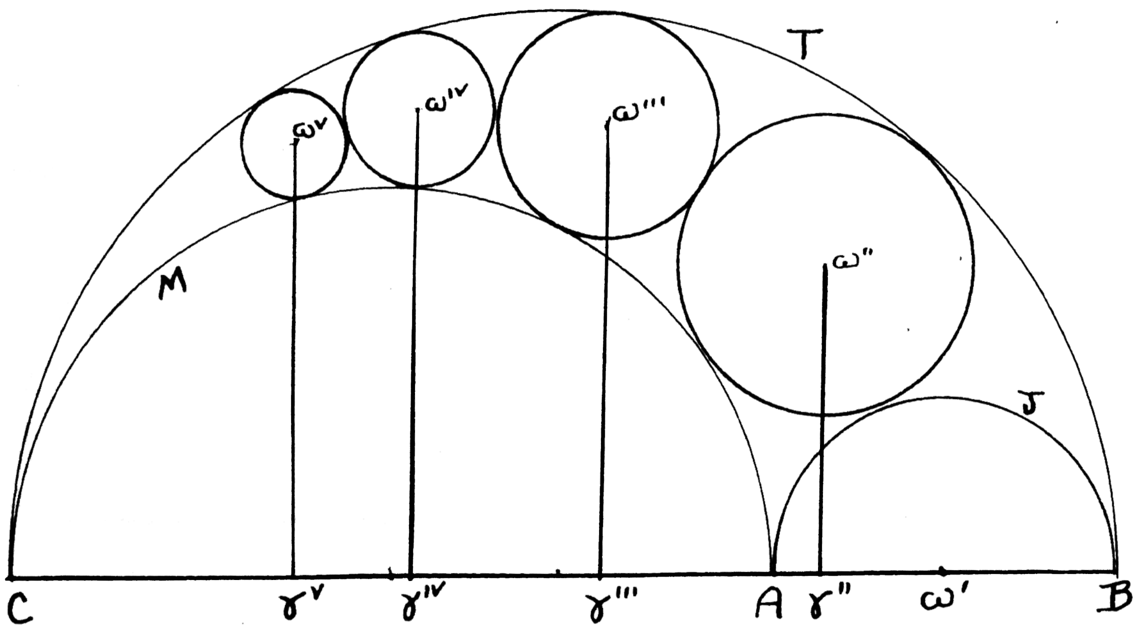

Figure 14

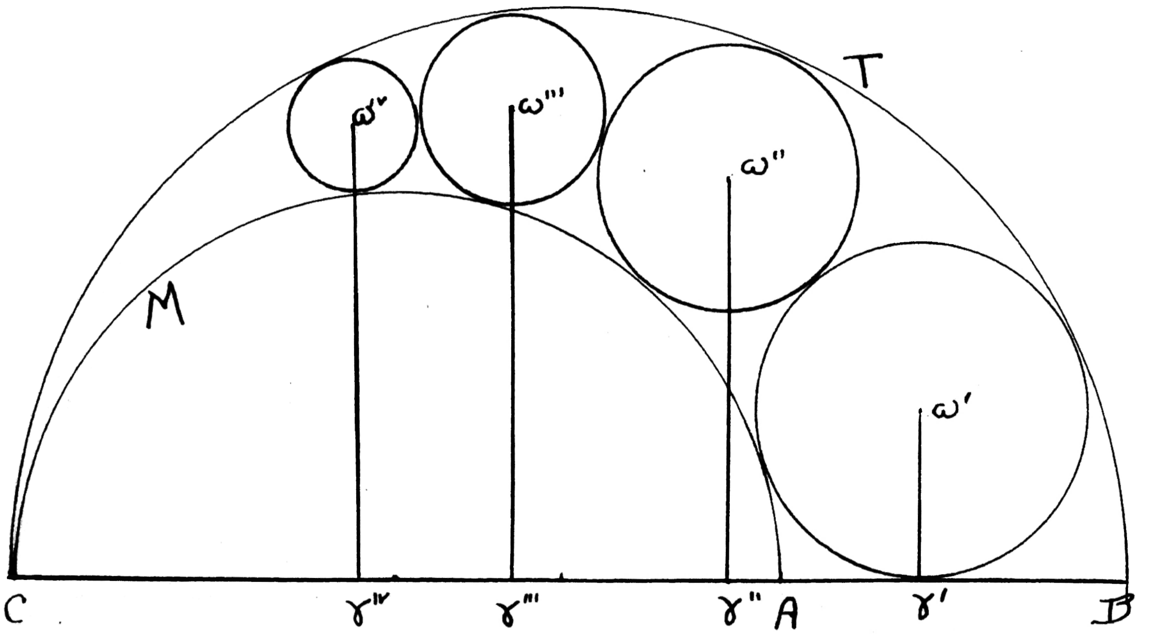

Figure 15

Corollary: The figure given for the theorem is susceptible of various modifications; for example, CJ′B may touch CM′A externally at C, and then circle ω' may touch both semi-circles externally or both internally. (Figure 13)

Whether CJ′B touches AM′C internally or externally, if its center moves off to infinity in either direction along CB, instead of the two semi-circles CJ′B, AM′C, there will be the straight line CE′ and the semi-circle AM′C, and the theorem will still be true.

M. SUCCESSIVE CIRCLES INSCRIBED IN THE SHOEMAKER'S KNIFE AND THE SHOEMAKER'S PASTEHORN. (Figures 14 and 15)

Let the semi-circles CTB and CMA be tangent to each other internally at C, and let a series of circles ω', ω'', ω''',--- whose radii are denoted by r_1, r_2, r_3, --- touch CTB and CMA and each other consecutively. If from the centers ω', ω'', ω''',--- perpendiculars $\omega'\gamma'$, $\omega''\gamma''$, $\omega'''\gamma'''$,--- are drawn to CB, then

(a) when the center of the first circle ω' lies on CB, (Figure 13)

$$\frac{\omega'\gamma'}{r_1}, \quad \frac{\omega''\gamma''}{r_2}, \quad \frac{\omega'''\gamma'''}{r_3}, \ldots\ldots, \quad \frac{\omega^n\gamma^n}{r_n}$$
$$= \quad 0, \quad 2, \quad 4, \ldots\ldots, \quad 2(n-1);$$

(b) when the first circle ω' touches CB, (Figure 14)

$$\frac{\omega'\gamma'}{r_1}, \quad \frac{\omega''\gamma''}{r_2}, \quad \frac{\omega'''\gamma'''}{r_3}, \ldots\ldots, \quad \frac{\omega^n\gamma^n}{r_n}$$
$$= \quad 1, \quad 3, \quad 5, \ldots\ldots, \quad 2n-1.$$

In other words, the quotients obtained in the manner above described from the Shoemaker's Knife are the even numbers; and those obtained from the Shoemaker's Pastehorn, as the other figure is called, are the odd numbers.*

For $$\frac{\omega'\gamma' + 2r_1}{r_1} = \frac{\omega''\gamma''}{r_2}$$

$$\therefore \frac{\omega'\gamma'}{r_1} + 2 = \frac{\omega''\gamma''}{r_2}.$$

Now when the center ω' lies on CB, $\omega'\gamma' = 0$, $\therefore \frac{\omega''\gamma''}{r_2} = 2.$

Again $$\frac{\omega''\gamma'' + 2r_2}{r_2} = \frac{\omega'''\gamma'''}{r_3}$$

$$\therefore \frac{\omega''\gamma''}{r_2} + 2 = \frac{\omega'''\gamma'''}{r_3}$$

when $\frac{\omega''\gamma''}{r_2} = 2$, $\frac{\omega'''\gamma'''}{r_3} = 4$, etc. ....

When the circle ω' touches CB, $\frac{\omega'\gamma'}{r_1} = 1$, hence

$\frac{\omega''\gamma''}{r_2} = 3$, $\frac{\omega'''\gamma'''}{r_3} = 5$ and so on.

Corollary: It will be seen that in these figures there are three circles CTB, CMA and ω' in mutual contact, and that of the series of circles ω'', ω''', ω^{IV}, ---, the first touches ω', the second ω'', and so on, while all touch CTB and CMA. If out of the three circles of mutual contact, instead of choosing CTB and CMA to be tangent to all of the

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* Pappus, Book IV, Props. 16, 18.

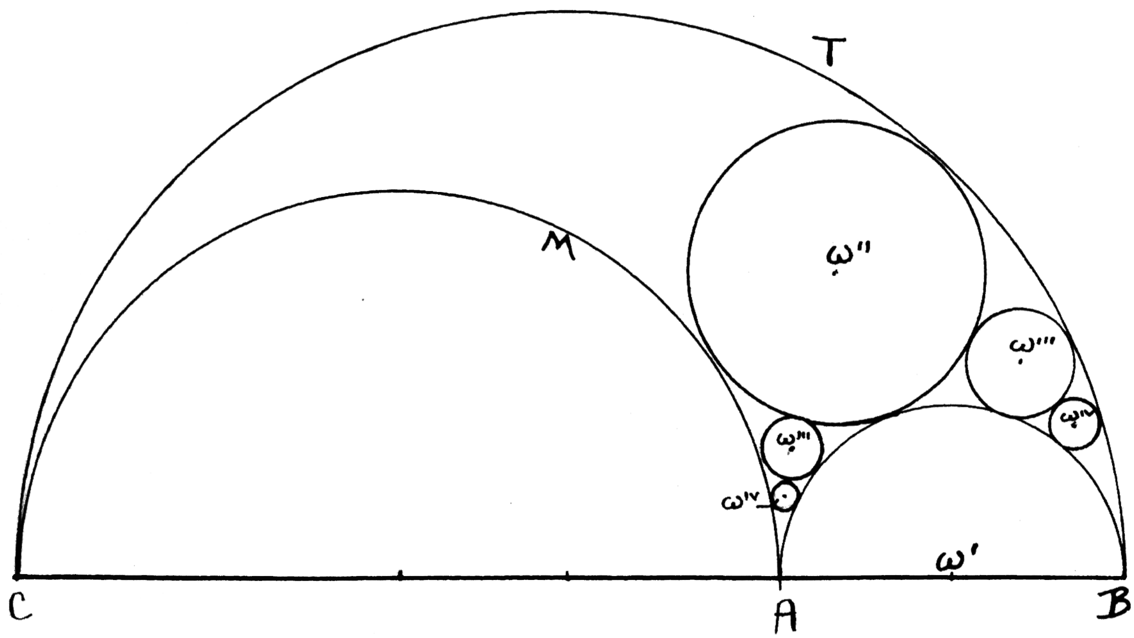

Figure 16

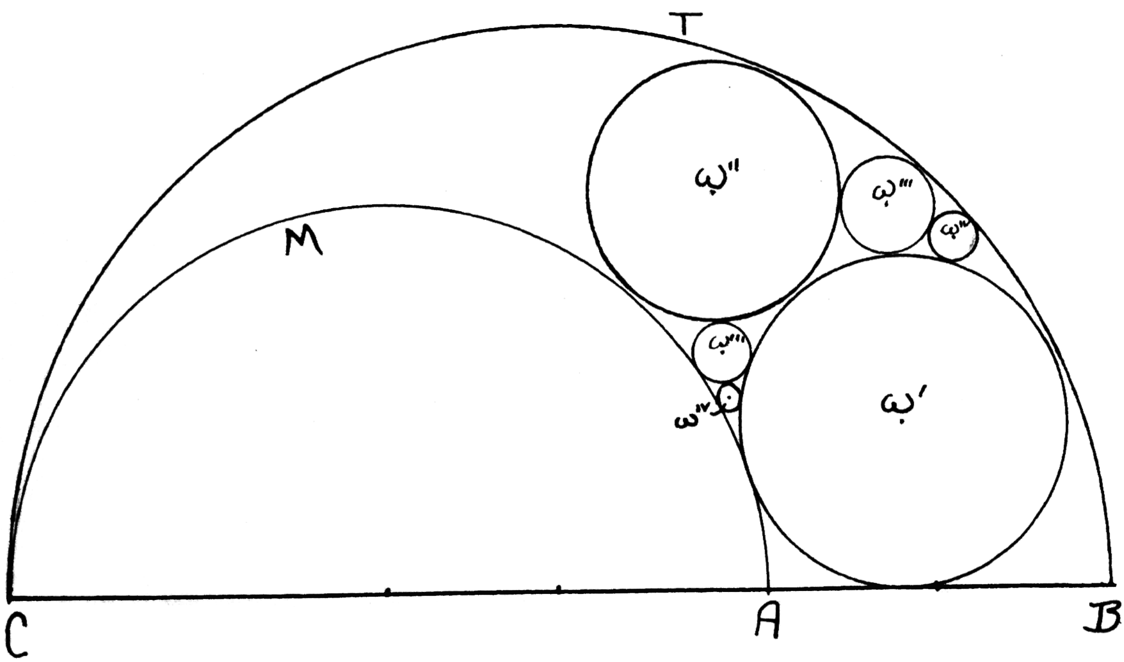

Figure 17

circles ω'', ω''', ω^{IV},---- we choose CTB and ω', we shall have ω'' as before, tangent to CMA, and a second series ω''', ω^{IV}, consecutively inscribed in the curvilinear space bounded by the circumferences ω', ω'' and CTB. (Figures 16 and 17)

If we choose CMA and ω' to be tangent to all the series ω'', ω''', ω^{IV}, ---- we shall have ω'' as before, tangent to CTB, and a third series ω''', ω^{IV}, ---- consecutively inscribed in the curvilinear space bounded by the circumferences ω', ω'', and CMA.

With respect to these two series of circles, the property of article M holds good. It also holds good with respect to the three series of circles that may be inscribed when the semi-circles CTB and CMA are replaced by straight lines perpendicular to CB at the points B and A; these straight lines being the limits toward which the two semi-circles tend when their centers move off to infinity in the direction of BC.

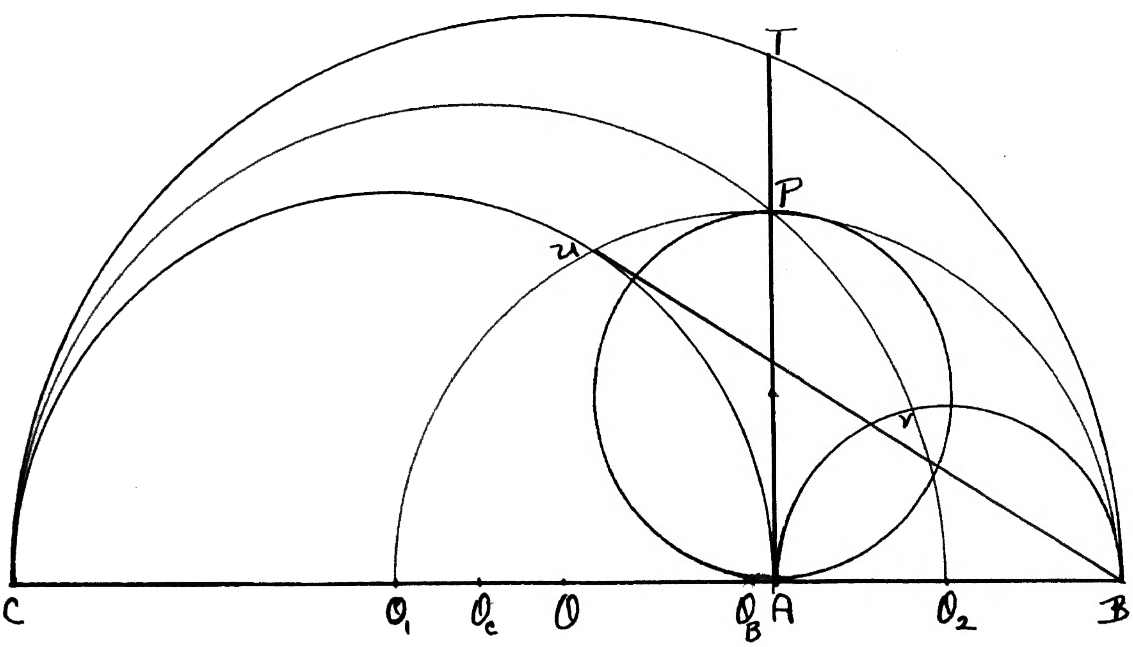

Figure 18

III. MODERN THEOREMS

A. LOCATION OF THE POINT OF INTERSECTION OF THE CIRCLES ON O_2C AND O_1B AS DIAMETERS. (Figure 18)

Let A, B and C be three points on a line such that A is between B and C. On the same side of the line, draw the semi-circles of diameter BC, CA, and AB. Let O be the center of the circle on BC of radius a, O_1 the center of the circle on CA of radius b, and O_2 the center of the circle on AB of radius c. Draw also the circles with diameters O_1B and O_2C whose centers are O_B and O_C respectively, and the common tangent at A to the circles AC and AB.

The point of intersection of circles O_1B and O_2C lies on the common tangent of circles CA and AB.*

Proof:

Let P_1 be the point of intersection of circles O_1B and the line AT, and let P_2 be the point of intersection of circle O_2C and line AT. The line AT is the locus of all points from which equal tangents can be drawn to the circles on AB and AC.

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* d'Ocagne, l'Enseigment Math., 1933, pp. 73-77.

$$\overline{AP_1}^2 = \overline{O_B P_1}^2 - \overline{AO_B}^2$$

$$= \left(\frac{b+2c}{2}\right)^2 - \left(2c - \frac{b+2c}{2}\right)^2$$

$$= \frac{b^2+4bc+4c^2}{4} - \frac{4c^2-4bc+b^2}{4}$$

$$= 2bc.$$

$$\overline{AP_2}^2 = \overline{O_C P_2}^2 - \overline{O_C A}^2$$

$$= \left(\frac{2b+c}{2}\right)^2 - \left(2b - \frac{2b+c}{2}\right)^2$$

$$= \frac{4b^2+4bc+c^2}{4} - \frac{4b^2-4bc+c^2}{4}$$

$$= 2bc.$$

Since AP_1 equals AP_2, the points P_1 and P_2 must coincide and hence the two circles cut the radical axis in the same point, P.

B. <u>THE AREA OF THE ARBELOS IS TWICE THE AREA OF THE CIRCLE ON AP AS DIAMETER</u>. (Figure 18)

Proof:

The area of the arbelos is

$$\frac{\pi}{2}\left(a^2 - b^2 - c^2\right)$$

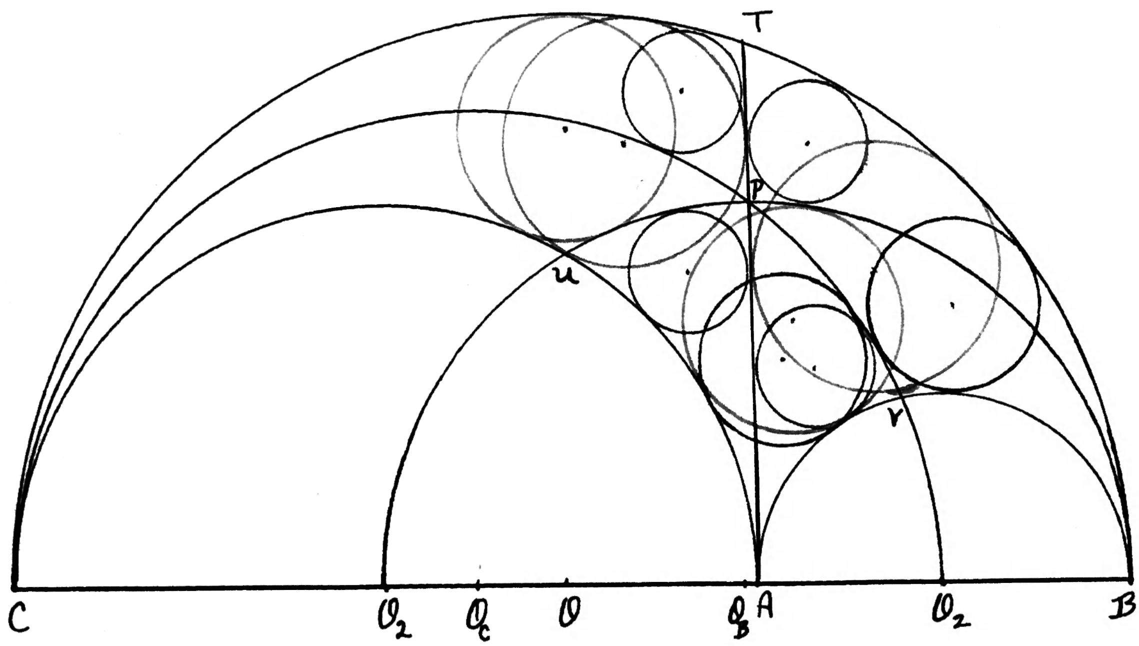

Figure 19

$$= \frac{\pi}{2}\left[(b+c)^2 - b^2 - c^2\right]$$

$$= \frac{\pi}{2}(2bc)$$

$$= \pi bc.$$

The area of the circle of diameter AP is

$$\pi\left(\frac{AP}{2}\right)^2$$

$$= \frac{\pi}{4}(AP)^2$$

$$= \frac{\pi}{4}(2bc)$$

$$= \frac{\pi bc}{2}.$$

Hence the area of the arbelos is twice the area of the circle on AP as diameter.

C. <u>EQUALITY OF CIRCLES INSCRIBED IN THE ARBELOS</u>.

In figure 19, circle O_1B cuts circle AC in the point of contact of the tangent from B to circle AC, and circle O_2C cuts circle AB in the point of contact of the tangent from C to circle AB. Let U be the point of intersection of circles on AC and O_1B and V be the point of intersection of the circles on AB and O_2C. Then U is the point of contact of the tangent from B to circle AC and V is the point of contact of the tangent from C to circle AB.

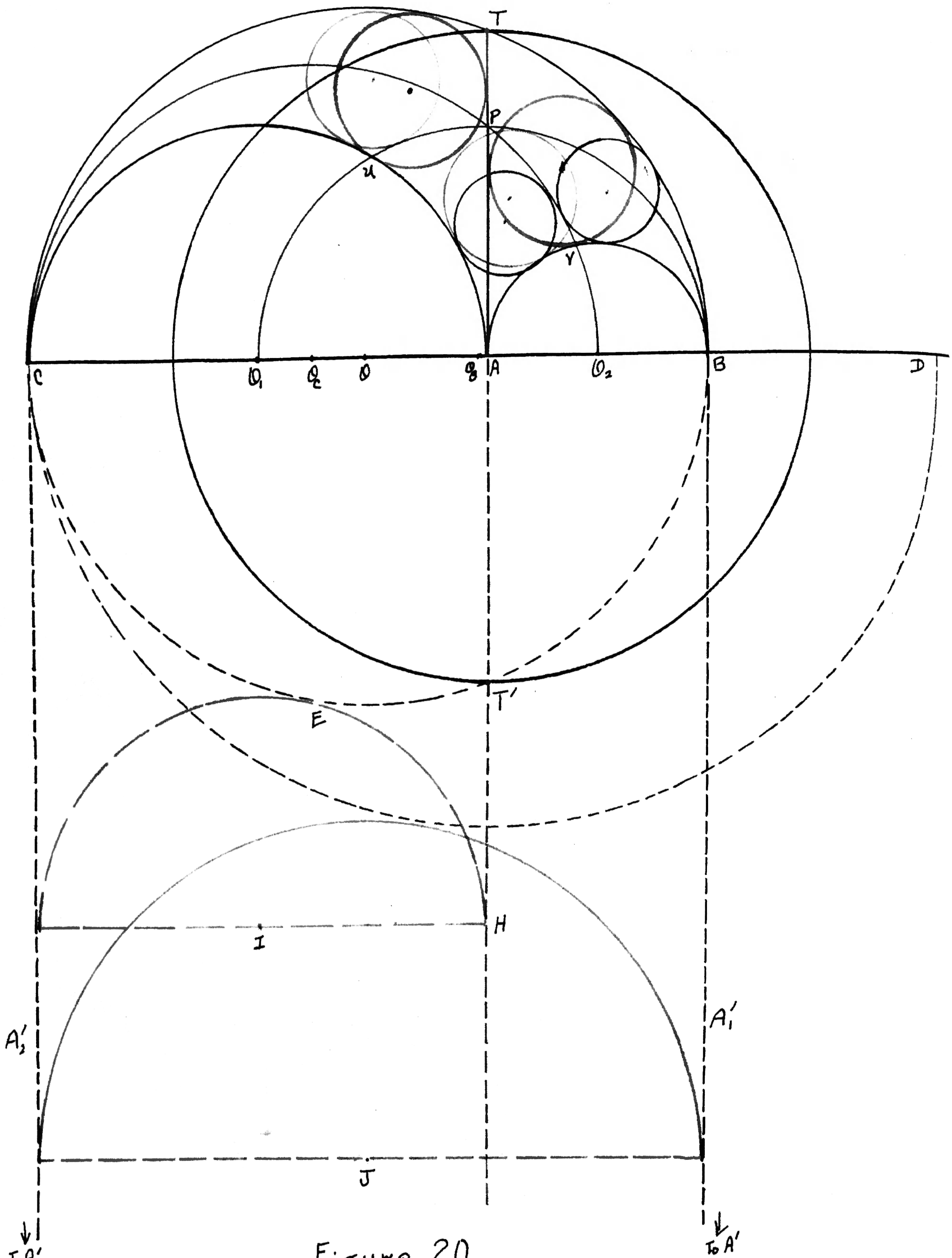

Figure 20

Thus the arbelos is divided into several sections. Taken in pairs, the circles inscribed in these sections are equal; i.e. in figure 19 the two green circles are equal, the two purple circles are equal, the two orange circles are equal, the four red circles are equal and the diameter of each red circle is half the size of the diameter of each orange circle. The proofs are given by inversion with respect to A.

1. Let A be the center of inversion with the power of inversion equal to

$$AB \cdot AC$$
$$= 2c \cdot 2b$$
$$= 4bc$$

in absolute value, and let the radii vectors be drawn directly opposite the usual sense since it will be easier to solve the problem that way than by the direct determination. (Figure 20)

Under this inversion the line AT is transformed into itself, but the part above CB now lies below CB; the circle BTC goes into itself, but the part above now lies below CB; and the semi-circles AC and AB go into lines through B anc C respectively, perpendicular to BC and below BC. The point C is the inverse of point B and point O_1 is the inverse of D. Therefore the inverse of semi-circle O_1B is a semi-circle on CD. The radius of semi-circle CD can be found by use of the formula for the ratio of the radii of a circle and its inverse which is

$$(1) \qquad \frac{R'}{R} = \frac{r^2}{\rho}$$

where r^2 is the square of the radius of inversion and ρ is the power of the center of inversion with respect to the circle of radius R.*

$$\frac{\text{radius of } O_1B}{\text{radius of } CD} = \frac{c}{a}$$

$$\frac{c + \frac{b}{2}}{O_BD} = \frac{2bc}{4bc}$$

$$2(c + \tfrac{b}{2}) = O_BD$$

$$2c + b = O_BD$$

$$c + b + c = O_BD \qquad (a = b + c)$$

$$a + c = O_BD$$

2. To determine the radius of the circle inscribed in segment ABT and in segment ACT of the arbelos. (Figure 20) Under the inversion with respect to A the point of tangency on AT is transformed into an inverse point on AT′; the point of tangency on $\widehat{BT}$ into an inverse point on $\widehat{CT'}$; and the point of tangency on $\widehat{AB}$ into an inverse point on CA'_2. Hence the radius of the inverse circle will be b. If C

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* Daus, College Geometry, p. 55.

represents the power of the pole A with respect to the circle with center I, the radius α of the circle inscribed in ABT is given by formula (1)

$$\frac{\alpha}{b} = \frac{4bc}{C}$$

$$\text{or} \quad \alpha = \frac{4b^2c}{\overline{AH}^2}$$

$$= \frac{4b^2c}{\overline{OI}^2 - \overline{OO_1}^2}$$

$$\text{but} \quad OI = a+b \quad \text{and} \quad OO_1 = O_1B - OB$$

$$= b + 2c - a$$

$$\therefore \quad C = (a+b)^2 - (b+2c-a)^2$$

$$= (a+b)^2 - [b+2(a-b)-a]^2$$

$$= (a+b)^2 - (a-b)^2$$

$$= 4ab$$

$$\therefore \quad \alpha = \frac{4b^2c}{4ab} = \frac{bc}{a}$$

Now consider the circle inscribed in segment ACT of the arbelos. Its inverse will be a circle tangent to AT′, BA_1' and arc BT′ having a radius of c. By using the same reasoning as above and interchanging b and c, the radius of the circle in segment ACT can be shown to be equal to $\frac{bc}{a}$.

Therefore the circle inscribed in ABT is equal to the circle inscribed in ACT. This is a new proof of the Ancient Theorem of acticle C, 1, section II.

3. To determine the radii of the circle in segment ABU and in segment BCU of the arbelos. (Figure 20) The point of tangency on $\overset{\frown}{AB}$ will go into a point on $A_1'C$, the point of tangency on $\overset{\frown}{AU}$ into a point on BA_1', and the point of tangency on $\overset{\frown}{UB}$ into a point on $\overset{\frown}{CD}$. The radius of the inverse circle will be a and the distance of the center from O_B is $a + O_BD = 2a + c$. Using formula (1) of article C, 1, to get the radius β of the original circle gives the equation,

$$\frac{\beta}{a} = \frac{G}{C} = \frac{4bc}{C}.$$

The power, C, of the pole A with respect to the circle of center J is

$$\begin{aligned}
\overline{AJ}^2 - a^2 &= \overline{OJ}^2 + \overline{OA}^2 - a^2 \\
&= \overline{O_BJ}^2 - \overline{OO_B}^2 + \overline{OA}^2 - a^2 \\
&= (2a+c)^2 - (a+c-a)^2 + (a-2c)^2 - a^2 \\
&= 4a^2 + 4ac + c^2 - c^2 + a^2 - 4ac + 4c^2 - a^2 \\
&= 4(a^2 + c^2)
\end{aligned}$$

$$\therefore \quad \beta = \frac{4abc}{4(a^2+c^2)} = \frac{abc}{a^2+c^2}$$

Now consider the circle inscribed in segment BCU of the arbelos. Its inverse will be a circle tangent to $\widehat{CD}$, $\widehat{T'B}$, and BA'. By using the same reasoning as above and interchanging a and c, we find that the radius of the circle in segment BCU is again $\dfrac{abc}{a^2+c^2}$.

4. Continuing to use this method, one can find for the radii of the circles inscribed in the segments ACV and CBV of the arbelos, $\gamma = \dfrac{abc}{a^2+b^2}$.

5. Consider now the three radii

$$\alpha = \frac{bc}{a} \qquad\qquad a^2\alpha = abc$$

$$\beta = \frac{abc}{a^2+c^2} \qquad\qquad (a^2+c^2)\beta = abc$$

$$\gamma = \frac{abc}{a^2+b^2} \qquad\qquad (a^2+b^2)\gamma = abc$$

$$\therefore\ a^2\alpha = (a^2+c^2)\beta = (a^2+b^2)\gamma = abc$$

$$\frac{1}{\gamma} = \frac{a}{bc}$$

$$= \frac{b+c}{bc}$$

$$= \frac{1}{b} + \frac{1}{c}$$

$$\frac{1}{\beta} - \frac{1}{\gamma} = \frac{a^2+c^2}{abc} - \frac{a^2+b^2}{abc}$$

$$= \frac{c^2-b^2}{abc}$$

$$= \frac{c}{ab} - \frac{b}{ac}$$

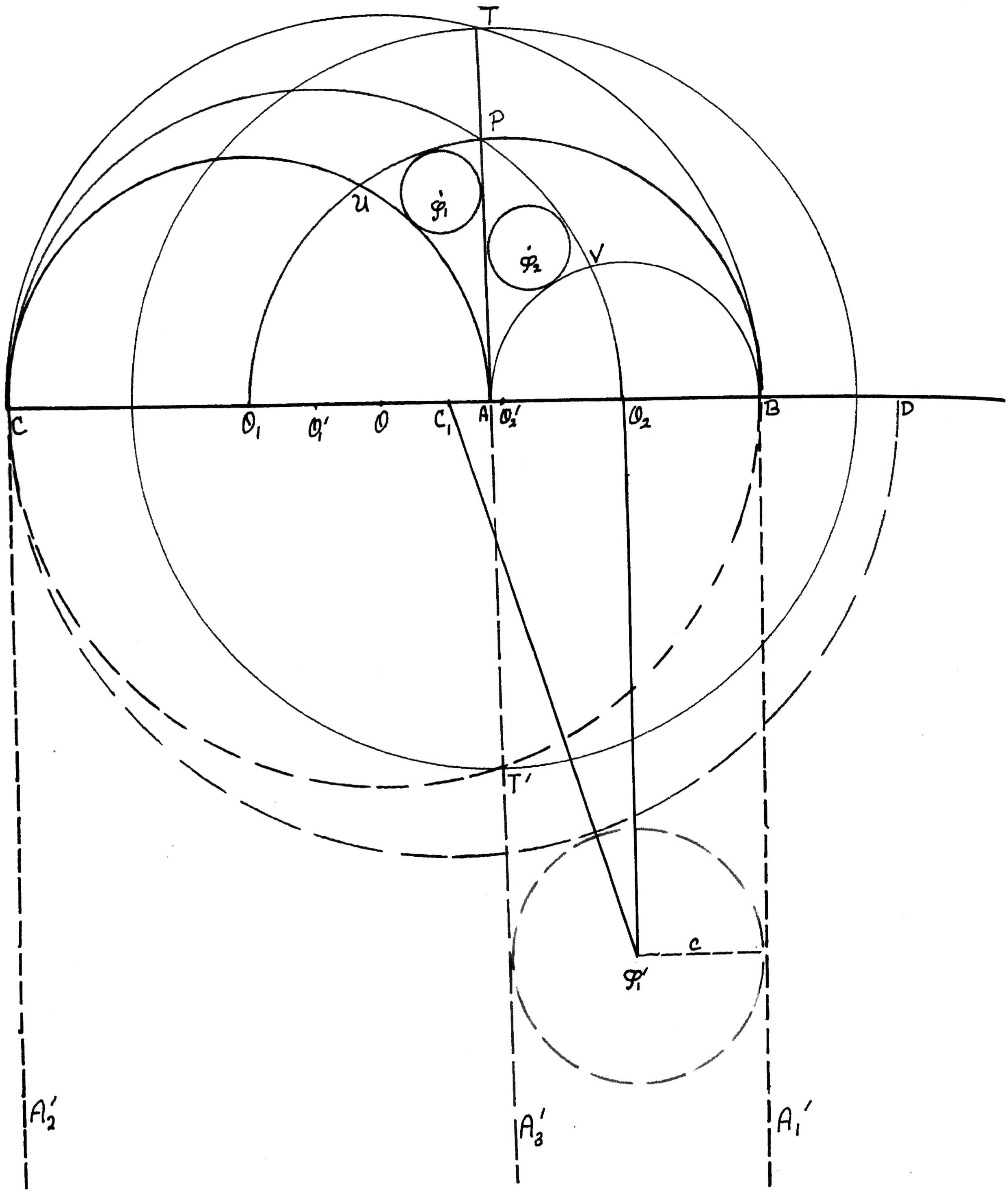

Figure 21

$$= \frac{1}{a}\left(\frac{c}{b} - \frac{b}{c}\right)$$

$$= \frac{1}{b+c}\left(\frac{c^2 - b^2}{bc}\right)$$

$$= \frac{c-b}{bc}$$

$$= \frac{1}{b} - \frac{1}{c}.$$

6. To determine the radii of the circles inscribed in segment AUP and segment AVP of the arbelos. (Figure 21) Under the inversion with respect to A the point of tangency on AT will go into an inverse point on AT′; the point of tangency on $\widehat{PU}$ into an inverse point on $\widehat{CD}$; and the point of tangency on $\widehat{AU}$ into an inverse point on BA_1'. Hence the radius of the inverse circle will be c. If C represents the power of the pole A with respect to the circle of center φ', the radius ξ of the circle inscribed in APU is given by formula (1)

$$\frac{\xi}{c} = \frac{4bc}{C}$$

$$\text{or} \quad \xi = \frac{4bc^2}{\overline{O_2\varphi_1'}^2} = \frac{4bc^2}{\overline{C_1\varphi_1'}^2 - \overline{C_1O_2}^2}.$$

But $C_1\varphi_1' = \tfrac{1}{2}CD + c = \tfrac{1}{2}(CA + AD) + c = \tfrac{1}{2}(2b + 4c) + c = b + 3c$

$$C_1O_2 = CO_2 - CC_1 = CO_2 - \tfrac{1}{2}CD = 2b + c - (b + 2c) = b - c$$

Therefore $\xi = \dfrac{4bc^2}{(b+3c)^2 - (b-c)^2} = \dfrac{bc}{2(b+c)} = \dfrac{bc}{2a}.$

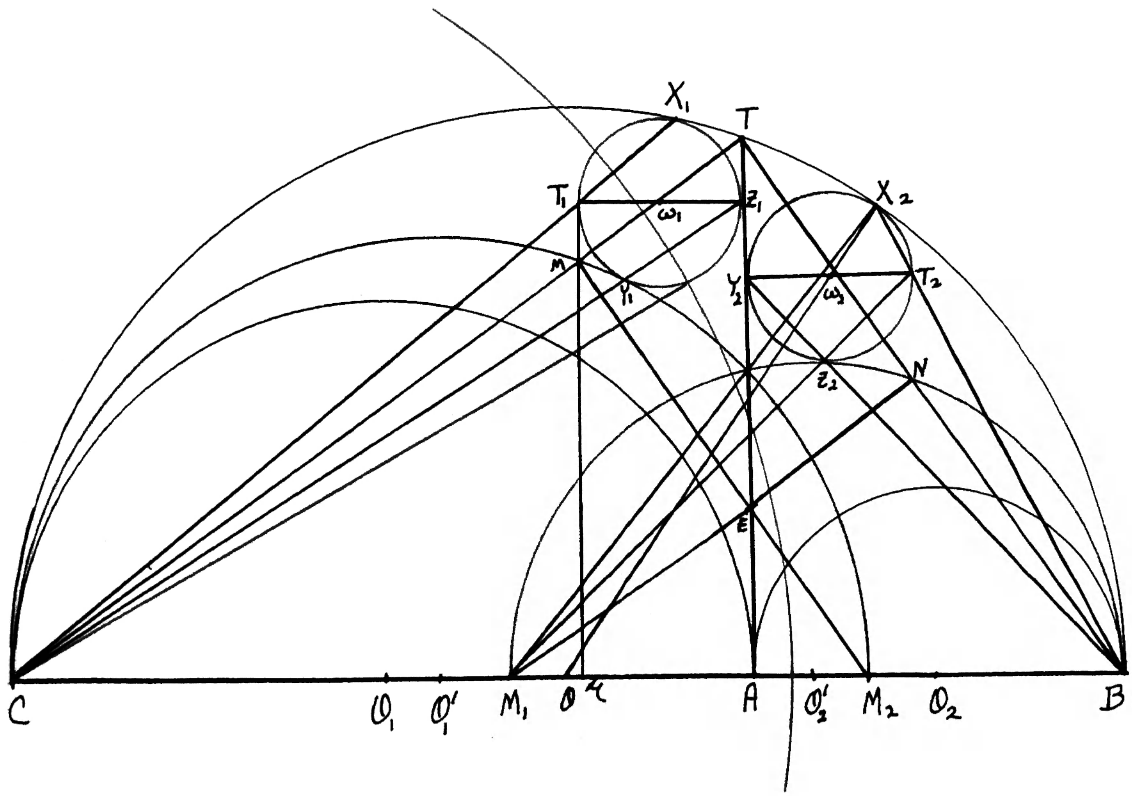

Figure 22

By the same method the radius of the circle inscribed in segment APV can be shown to be $\frac{bc}{2a}$; hence the two circles are equal.

7. By using the same method again, the radii of the circles inscribed in segments PCT and PBT of the arbelos can be shown to equal $\frac{bc}{2a}$.

8. Therefore, by use of the results in (2), (6) and (7), the diameters of circles inscribed in the segments PCT, PBT, AUP, AVP are equal to the radii of the circles inscribed in the segments ACT and ABT of the arbelos.

9. Another method of proving the pair of circles inscribed in segments PCT and PBT equal is given here.* (Figure 22) Consider the points M_1 and M_2 into which C and B respectively are transformed by the homothetic transformation:

$$\frac{AP'}{AP} = k$$

Above BC draw the semi-circles O_2', O_1' with BM_1 and CM_2 respectively, as diameters. Through A and the point of intersection of the semi-circles O_2' and O_1' , draw a line which meets the semi-circle O in T. Then draw the circles ω_1 and ω_2 tangent to the semi-circle O, to the line AT and

* Thebault, l'Enseignement Mathematique, 1934, p. 349.

to the semi-circles O_1' and O_2' respectively. Let X_1, Z_1, Y_1 and X_2, Z_2, Y_2 be the respective points of contact. The line AT is the radical axis of circles O_1' and O_2' for:

$$|AM_1 \cdot AB| = |k \cdot AC \cdot AB| = |AM_2 \cdot AC|.$$

Moreover since X_1 is the external homothetic center of circles ω_1 and O, the line CX_1 meets ω_1 in point T_1, the point of contact of the line which is tangent to the circle and is also perpendicular to BC. The line CY_1 cuts the circle ω_1 in the point of contact of that circle with a tangent perpendicular to BC since Y_1 is the internal homothetic center of ω_1 and O_1' ; that is, it cuts AT in Z_1. Let M and N be the points of intersection of the lines CT and BT with the half-circles O_1' and O_2' ; let η be the projection of the point M on BC.

Transform the figure by an inversion with center C and the square of radius of inversion equal to

$$\begin{aligned} CM \cdot CT &= CY_1 \cdot CZ_1 \\ &= CT_1 \cdot CX_1 \\ &= C\eta \cdot CB \qquad (\triangle CM\eta \sim \triangle CBT) \end{aligned}$$

Under this inversion the point T_1 goes into X_1, Y_1 into Z_1 and the points of tangency K_i into themselves and hence the circle ω_1 is transformed into itself. Also under this inversion, M goes into T, η into B, and T_1

into X_1, hence the circle O is transformed into the line T_1M and consequently the line T_1M goes through μ .

The circle ω_1 is thus seen to be inscribed in a mixtilinear trapezium bounded by two straight lines, AT and μT_1, and two arcs, O_1' and O. Since AT is parallel to MT_1, the diameter of ω_1 is T_1Z_1, which is equal to $2\rho_1$. Similarly circle ω_2 is inscribed in a mixtilinear trapezium formed by the arcs O and O_2' and the line AT and the line through N perpendicular to AB. The diameter is Y_2T_2 which is equal to $2\rho_2$.

It is desired to show that T_1Z_1 equals Y_2T_2. Let E be the point of intersection of MM_2 and NM_1. Since T is on the radical axis of semi-circles O_1' and O_2', it follows that

$$CT \cdot TM = TB \cdot TN$$

$$\text{or } \frac{CT}{TB} = \frac{TN}{TM} = \frac{ME}{NE}.$$

Since M_1N is perpendicular to BT and M_2M is perpendicular to CT,

$$\triangle M_1EM_2 \sim \triangle CTB.$$

$$\therefore \frac{CT}{TB} = \frac{M_1E}{M_2E}$$

$$\frac{ME}{NE} = \frac{M_1E}{M_2E}$$

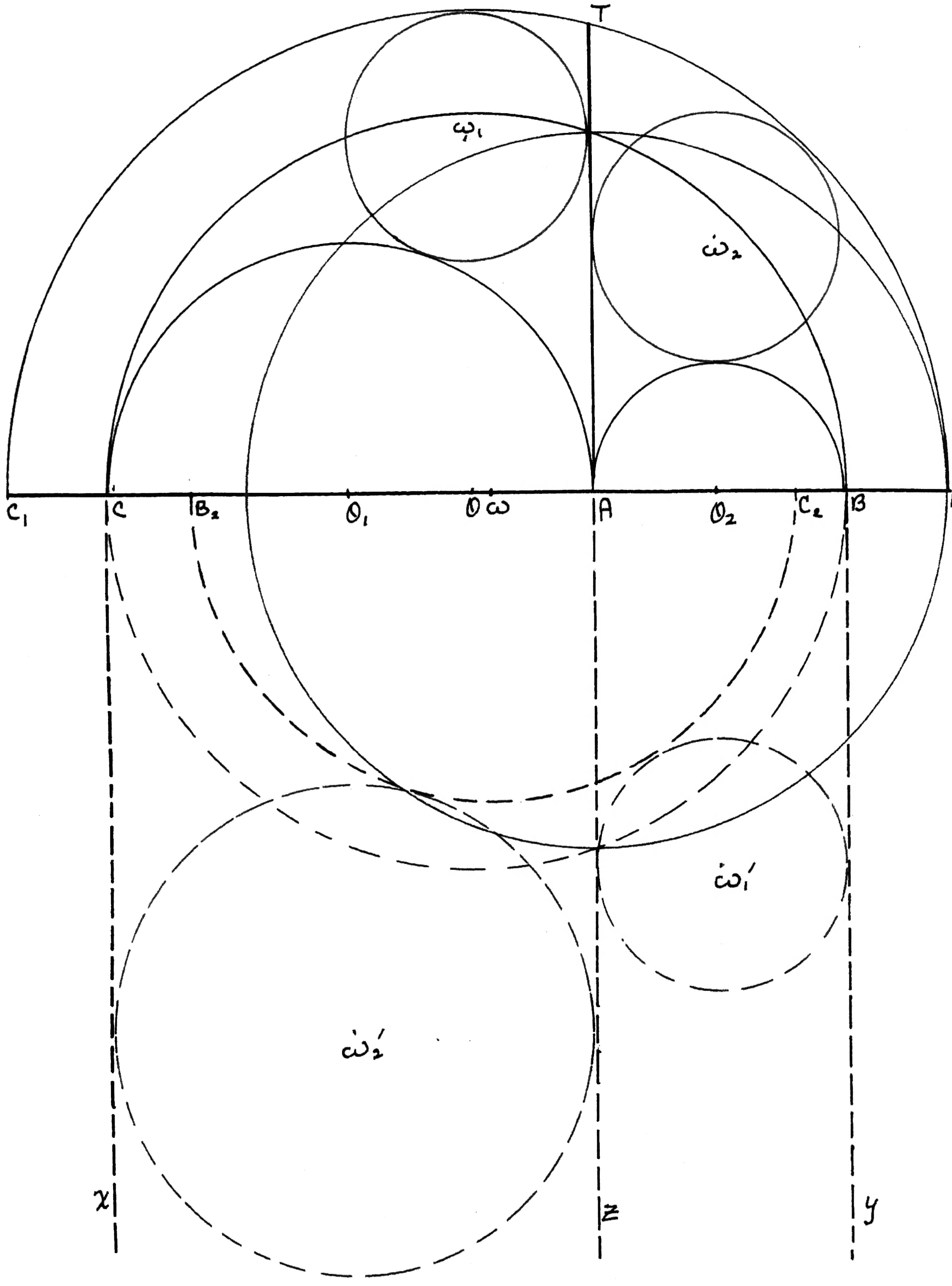

Figure 23

or $ME \cdot M_2E = NE \cdot M_1E$.

Therefore E is on the radical axis.

Then MN and TE are equal, since they are diagonals of the rectangle MENT. The diagonals of a rectangle bisect each other, hence ML equals LN. The horizontal projection of ML is T_1Z_1 and of LN is T_2Y_2; hence T_1Z_1 equals Y_2T_2.

D. <u>THE INSCRIBED CIRCLES ARE ALSO EQUAL IF CIRCLES O_1 AND O_2 DO NOT TOUCH CIRCLE O.</u> (Figure 23)

Theorem: <u>If we replace the semi-circle O drawn on BC as diameter by a semi-circle concentric with the first, keeping fixed the two semi-circles O_1 and O_2 with AC and AB respectively as diameters, then the radii of the circles ω_1 and ω_2 which are tangent to the last semi-circle of center O_1 to the line perpendicular to BC at A and to the circles O_1 and O_2 respectively, are equal.</u>

Proof: Let BC = 2a, and the center of BC be O;
let AC = 2b, and the center of AC be O_1; and
let AB = 2c, and the center of AB be O_2.

Let d be a given distance and draw a circle with its center at O and its radius $R = b + c + d$. Let the semi-circle B_1C_1 be cut by the common tangent of circles O_1 and O_2 at A in point T. Draw circles ω_1 and ω_2 so that ω_1 is tangent to line AT, circle B_1TC_1 and circle O_1; ω_2 is tangent to line AT, circle B_1TC_1 and circle O_2. Choose A as the center of inversion with AB·AC as the power of inversion and the reciprocal radii drawn in opposite direction. The line AT goes into itself, but the upper part inverts into the part below the line BC. The circle AB inverts into a line, perpendicular to BC through C, the part below BC corresdponding to the arc above. The circle AC inverts into a line perpendicular to BC through B, the part below corresponding to the arc above. Since B goes into C and C into B, the circle O inverts into itself, the part above BC corresponding to the part below. The semi-circle with center O and radius R inverts into $\widehat{B_2C_2}$ below AB. Let ω be the center of semi-circle B_2C_2.

$$AC_2 \cdot |AC_1| = AB \cdot AC \qquad AB_2 \cdot |AB_1| = AB \cdot AC$$

$$AC_2 = \frac{4bc}{2b+d} \qquad AB_2 = \frac{4bc}{2c+d} \quad (2)$$

The formula for the ratio of the radii of the circle and its inverse is:*

$$\frac{R'}{R} = \frac{r^2}{p}$$

where r^2 is the square of the radius of inversion and p is the power of the center of inversion with respect to the circle of radius R.

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* Daus, College Geometry, p. 55.

$$p = \overline{AT}^2$$
$$= C_1A \cdot AB_1$$
$$= (R + AO)(R - AO)$$
$$= R^2 - \overline{AO}^2.$$

Let ρ be the radius of the circle that is the inverse of the circle with center O and radius R.

$$\frac{\rho}{R} = \frac{AB \cdot AC}{R^2 - \overline{AO}^2} = \frac{4bc}{(b+c+d)^2 - (b-c)^2}$$

$$\rho = \frac{4bc\,R}{d^2 + 4bc + 2cd + 2bd}$$

$$= \frac{4bc\,(b+c+d)}{d(d+2b) + 2c\,(d+2b)}$$

$$= \frac{4bc\,(b+c+d)}{(d+2b)\,(d+2c)} \qquad (3)$$

Under this inversion, ω_1 goes into a circle ω_1' tangent to line Az, line By and circle B_2C_2. The radius of ω_1' is c, and the radius of ω_2' which is tangent to lines Cx, Az, and circle B_1C_2 is b. Let ρ_1 and ρ_2 be the radii of the circles inverse to ω_1' and ω_2' respectively.

By the same formula which was used above, we find that

$$\frac{\rho_1}{c} = \frac{|AB \cdot AC|}{p_1} \qquad \text{and} \qquad \frac{\rho_2}{b} = \frac{|AB \cdot AC|}{p_2}$$

where p_1 and p_2 are the power of **A** with respect to circles ω_1' and ω_2' respectively.

Then

$$\frac{p_1}{c} = \frac{4bc}{\overline{A\omega_1'}^2 - c^2}$$

$$= \frac{4bc}{\overline{O_2\omega_1'}^2}$$

But $\overline{O_2\omega_1'}^2 = \overline{\omega\omega_1'}^2 - \overline{\omega O_2}^2 = (\rho + c)^2 - \overline{\omega O_2}^2$,

and $\omega O_2 = \omega A + AO_2$.

By using (2) and (3), we find the following value for ωA:

$$\omega A = \rho - AC_2 = \rho - \frac{4bc}{2b+d}$$

$$= \rho - \frac{\rho(2c+d)}{R}$$

$$= \frac{\rho}{R}\left[R - (2c+d)\right]$$

$$= \frac{\rho}{R}\left[(b+c+d) - (2c+d)\right]$$

$$= \frac{\rho}{R}(b-c)$$

Then $\overline{O_2\omega_1'}^2 = (\rho+c)^2 - \left[\frac{\rho}{R}(b-c) + c\right]^2$

$$= \rho^2 + 2\rho c + c^2 - \left[\frac{\rho^2(b^2-2bc+c^2)}{R^2} + \frac{2\rho c(b-c)}{R} + c^2\right]$$

$$= \rho^2\left[1 - \frac{(b-c)^2}{R^2}\right] + 2\rho c\left[1 - \frac{b-c}{R}\right]$$

$$= \rho\left[1 - \frac{b-c}{R}\right]\left[\rho\left(1 + \frac{b-c}{R}\right) + 2c\right]$$

$$= \frac{\rho}{R}(2c+d)\left[\frac{\rho(2b+d)}{R} + 2c\right]$$

$$= \frac{4bc}{(2b+d)(2c+d)}(2c+d)\left[\frac{4bc(2b+d)}{(2b+d)(2c+d)} + 2c\right]$$

$$= \frac{8bc^2}{2b+d} \cdot \frac{2b+2c+d}{2c+d}$$

$$\therefore \rho_1 = \tfrac{1}{2}\left[\frac{(2b+d)(2c+d)}{2b+2c+d}\right] \qquad (4)$$

If instead of ω_1, the circle ω_2 is considered, it is sufficient to interchange the parts played by semi-circles O_1 and O_2, namely to interchange b and c, thus showing that ρ_1 and ρ_2 are equal. Therefore $\omega_1 = \omega_2$.

If d = 0, the formula for ρ_1 and ρ_2 reduces to $\frac{bc}{a}$, as shown before.

Also formula (4) can easily be changed to the form

$$\frac{1}{2\rho_2} = \frac{1}{2\rho_1} = \frac{1}{2b+d} + \frac{1}{2c+d} - \frac{d}{(2c+d)(2b+d)}$$

$$\text{If } d=0, \quad \frac{1}{\rho_1} = \frac{1}{b} + \frac{1}{c} = \frac{1}{\rho_2}.$$

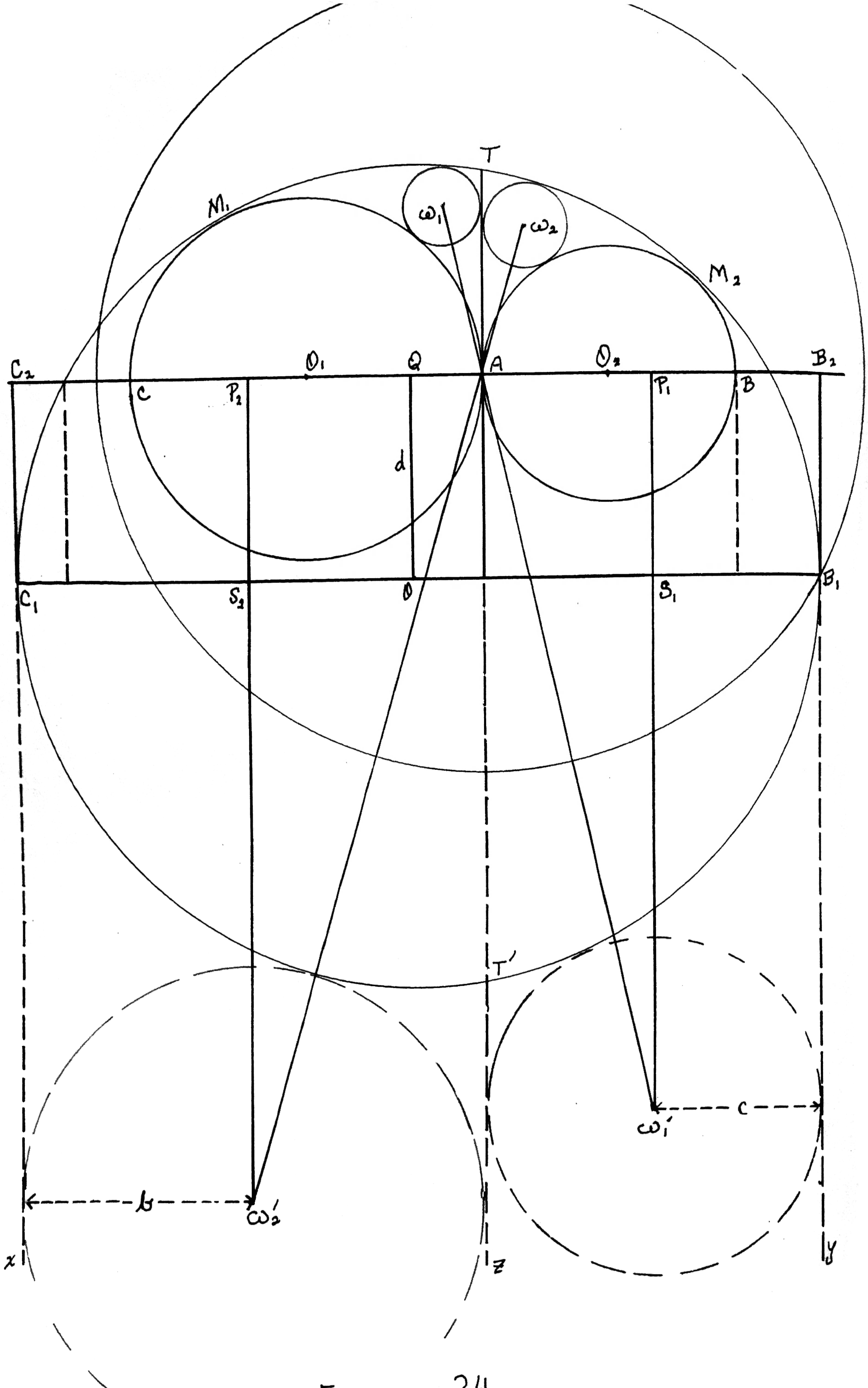

Figure 24

E. EQUALITY OF INSCRIBED CIRCLES WHEN THE DIAMETERS OF THE SMALL SEMI-CIRCLES ARE ABOVE THE DIAMETER OF THE LARGE SEMI-CIRCLE. (Figure 24.)

Theorem: Let two circles O_1 and O_2 of radii R_1 and R_2 be tangent externally at a point A and let a third circle O of radius R be tangent to those circles at M_1 and M_2 respectively, and let O contain O_1 and O_2. Then the circles inscribed in AM_1T and AM_2T are equal.

Draw the diameter B_1C_1 of circle O parallel to the line of centers O_1O_2. The line of centers cuts the circles O_1 and O_2 in C and B respectively, and cuts the lines through B_1 and C_1 perpendicular to BC, in B_2 and C_2 respectively. The perpendicular to BC at A cuts circle O in T and T$'$.

$$
\begin{aligned}
\text{Assume}\quad AC_2 &= 2b\\
\text{and}\quad AB_2 &= 2c\\
\text{then}\quad C_1B_1 &= 2(b+c),\\
\text{or}\quad R &= b+c.
\end{aligned}
$$

Drop a perpendicular from O to BC and call the foot of it Q. Let $OQ = d$, and let $b - c = d'$.

$$
\begin{aligned}
\text{Then}\quad OB_1 &= b+c\\
OB_1 - AB_2 &= QA,\\
\text{and}\quad QA &= b+c - AB_2 = b+c-2c = b-c.\\
\therefore\quad \text{from } O \text{ to } TAT' &= d'.
\end{aligned}
$$

Let ω_1 of radius ρ_1 and ω_2 of radius ρ_2 be the circles inscribed in mixtilinear triangles ATM_1 and ATM_2. Take A as the center of inversion and the square of radius of inversion

$AT \cdot AT'$

$= \overline{AO}^2 - R^2$ (negative since A is inside O.)

$= d^2 + (d')^2 - (b+c)^2$

$= d^2 + (b - c)^2 - (b+c)^2$

$= d^2 - 4bc.$

Draw the radii vectors in the opposite direction. The line AT is transformed into itself, but T becomes T′ and T′ becomes T. Circle O inverts into itself for T goes into T′ and the two points on the circle of inversion go into themselves, but the sides are interchanged. The circles O_1 and O_2 go into lines B_2B_1y and C_2C_1x respectively, since the following computation of the products $AC \cdot AB_2$ and $AB \cdot AC_2$ gives $4bc-d^2$.

$$AC = 2R_1$$

$$R_1 = OM_1 - OO_1 \qquad\qquad R_2 = OM_2 - OO_2$$

$$= R - \sqrt{d^2 + (R_1 - d')^2} \qquad\qquad = R - \sqrt{d^2 + (R_2 + d')^2}$$

$$\sqrt{d^2 + (R_1 - d')^2} = R - R_1 \qquad\qquad \sqrt{d^2 + (R_2 + d')^2} = R - R_2$$

$$d^2 + R_1^2 - 2R_1 d' + d'^2 = R^2 - 2RR_1 + R_1^2 \qquad\qquad d^2 + R_2^2 + 2R_2 d' + d'^2 = R^2 - 2RR_2 + R_2^2$$

$$2RR_1 - 2R_1 d' = R^2 - d^2 - d'^2$$

$$2R_1(R - d') = R^2 - d^2 - d'^2 \qquad\qquad 2R_2(R + d') = R^2 - d^2 - d'^2$$

$$R_1 = \frac{(b+c)^2 - d^2 - (b-c)^2}{2[(b+c) - (b-c)]} \qquad\qquad R_2 = \frac{(b+c)^2 - d^2 - (b-c)^2}{2(b - c + b + c)}$$

$$R_1 = \frac{4bc - d^2}{2 \cdot 2c} \qquad R_2 = \frac{4bc - d^2}{4b}$$

$$2R_1 = \frac{4bc - d^2}{2c} = AC \qquad 2R_2 = \frac{4bc - d^2}{2b} = AB$$

$$2R_1 \cdot 2c = 4bc - d^2 \qquad 2R_2 \cdot 2b = 4bc - d^2$$

$$AC \cdot AB_2 = 4bc - d^2 \qquad AB \cdot AC_2 = 4bc - d^2$$

Then circle ω_1 tangent to circles O, O_1 and line AT, becomes circle ω_1' tangent to circle O, and lines B_1y and TZ; and circle ω_2, tangent to circles O, O_2 and line AT, becomes circle ω_2' tangent to circle O, and lines C_1x and TZ. Consequently circle ω_1' has radius c and circle ω_2' has radius b.

In order to find the values of the radii ρ_1 and ρ_2, use the formula for the radius of inversion.*

$$\frac{\rho_1}{c} = \frac{|d^2 - 4bc|}{\overline{AH_1}^2} = \frac{|d^2 - 4bc|}{(T_1S_1 + S_1\omega_1')^2}$$

$$\overline{S_1\omega_1'}^2 = \overline{O\omega_1'}^2 - \overline{OS_1}^2$$

$$= (R + c)^2 - (d' + c)^2$$

$$= (b + 2c)^2 - (b)^2$$

$$= 4bc + 4c^2$$

$$= 4cR$$

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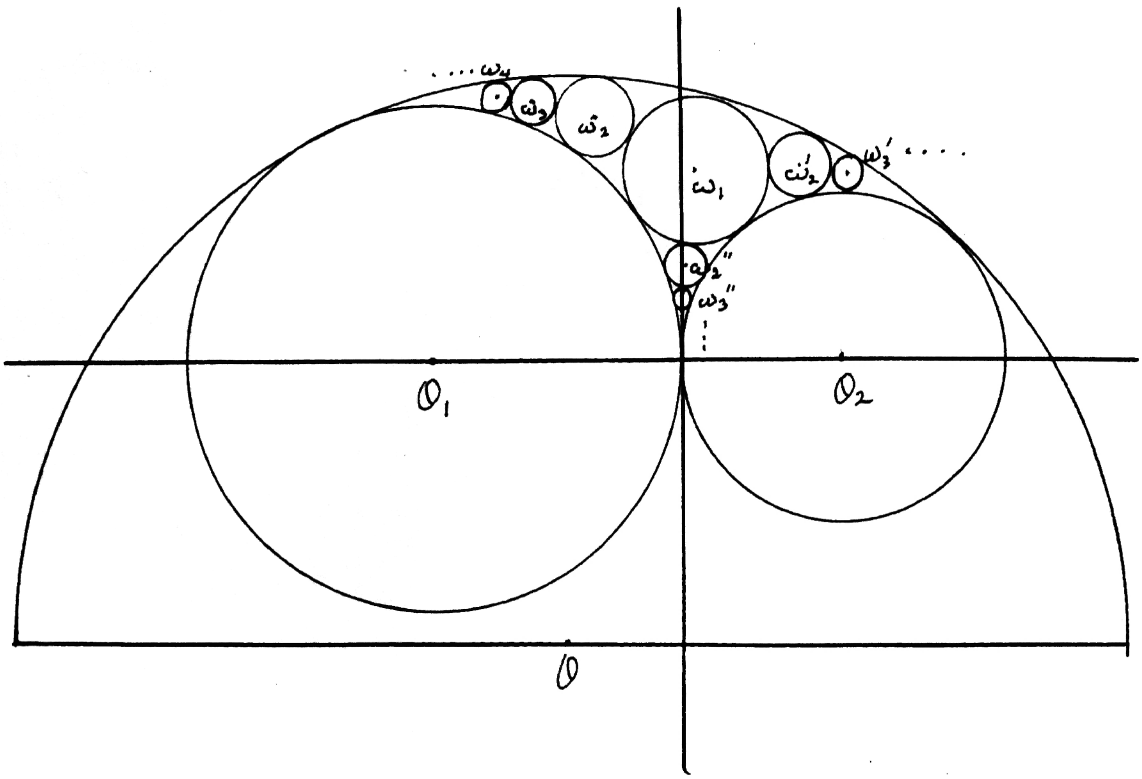
* Daus, College Geometry, p. 55.



ω_4
ω_3
ω_2
ω_1
ω_2'
ω_3'
ω_2''
ω_3''
O_1
O_2
O

Figure 25

$$\text{Then } \frac{\rho_1}{c} = \frac{|d^2 - 4bc|}{d^2 + 2d\sqrt{4cR} + 4cR}$$

$$= \frac{|d^2 - 4bc|}{d^2 + 4cR + 4d\sqrt{cR}}$$

$$\text{or } \rho_1 = \frac{c|d^2 - 4bc|}{d^2 + 4cR + 4d\sqrt{cR}}.$$

For circle ω_2 interchange b and c,

$$\rho_2 = \frac{b|d^2 - 4bc|}{d^2 + 4bR + 4d\sqrt{bR}}$$

$$\text{Then } \frac{\rho_1}{\rho_2} = \frac{c(d^2 + 4bR + 4d\sqrt{bR})}{b(d^2 + 4cR + 4d\sqrt{cR}}.$$

When $d = 0$, $b = R_1$, $c = R_2$ and $\rho_1 = \rho_2$.

This theorem is the theorem of article C of the section on the Ancient Theorems. V. Thebault has shown that this method of proof may be extended to the case in which the circle O may be replaced by a circle concentric with O of radius R+m. In this extended case the common tangent of circles O_1 and O_2 becomes the radical axis of those circles.*

F. <u>STUDIES OF THREE SEQUENCES OF CIRCLES</u>.

In Figure 25, let the ordinates of the centers

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* Thebault, l'Enseignement Mathematique, 1934, p. 352.

of the sequence of circles ω_i be y_i, and those of the sequence ω_i' be y_i', and those of the sequence ω_i'' be y_i'', and let the radii of these sequences be ρ_i, ρ_i' and ρ_i''. By use of the method of inversion with **A** as center and $d^2 - 4bc$ the power of the inversion Thebault* has established the following formulas for the ordinates of the centers of these sequences of circles:

$$y_n = 2n\rho_n\left(1 + \frac{nd}{2R}\right)$$

$$y_n' = 2n\rho_n'\left(1 + \frac{d}{2R}\right)$$

$$y_n'' = 2\rho_n''\left(n + \frac{d}{2R}\right)$$

These results are a generalization of the theorem given in Article M, section II.

From these results relations among the reciprocals of the radii similar to those obtained in Article M of the Ancient Theorem section may be found. These results were also established by M. G. Fontené in the Nouvelles Annales de Mathématiques, 1918, pp. 383-390, but his method of proof was different.

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* l'Enseignement Mathematique, Vol. 34, 1935, pp. 313-320.

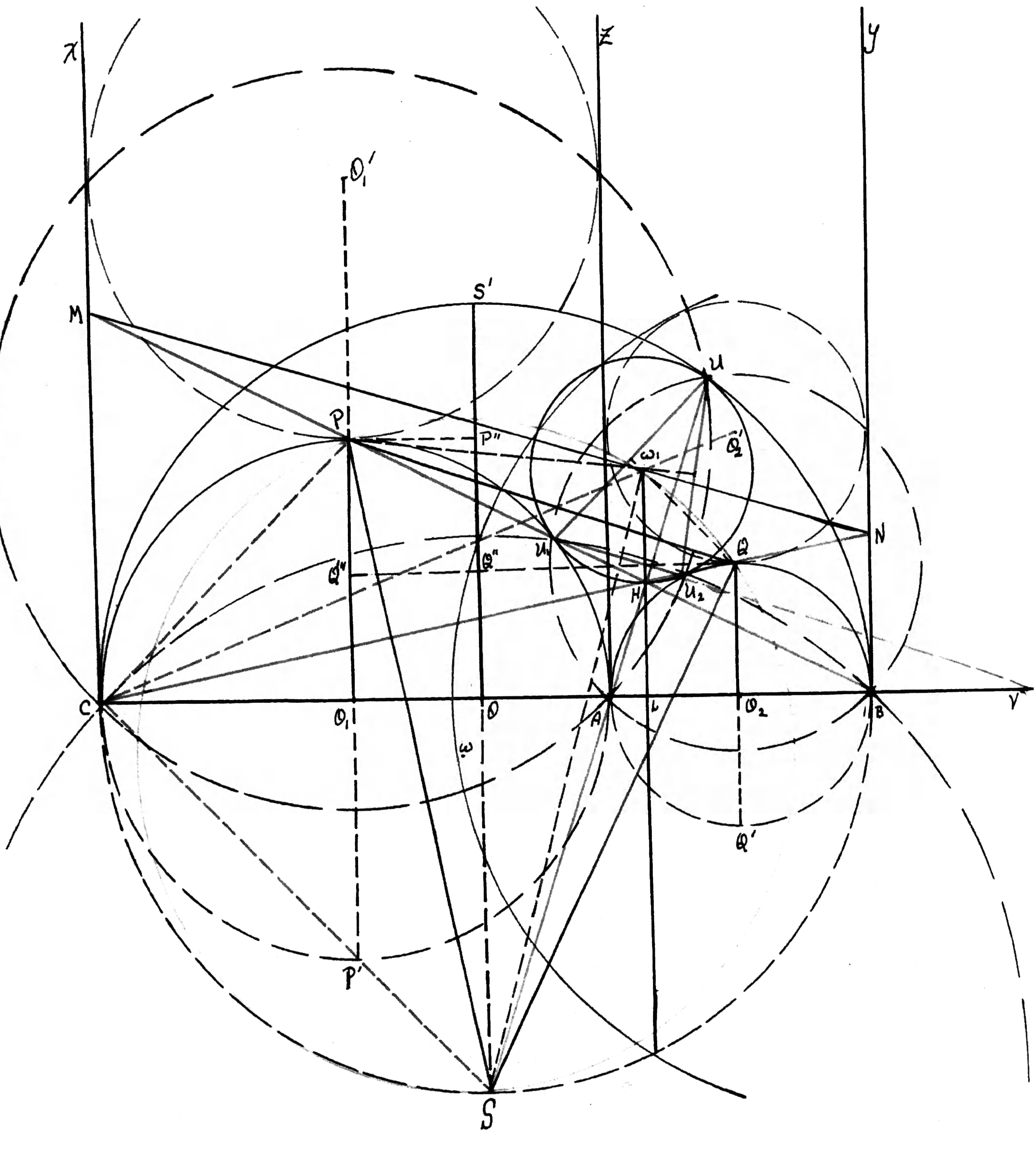

Figure 26

G. A STUDY OF THE PROPERTIES OF THE ARBELOS AND ITS INSCRIBED CIRCLE AND TRIANGLES. (Figure 26)

Given:

The arbelos ABC made up of the semi-circle on CB as diameter, with center O and radius equal to a; and the semi-circle on CA as diameter with center O_1 and radius equal to b; and the semi-circle on AB as diameter with center O_2 and radius equal to c. Let the mid-point of $\widehat{AC}$ be P, the point of symmetry with respect to AC be P'; let the mid-point of $\widehat{AB}$ be Q, the point of symmetry with respect to CB be Q'; and let the mid-point of $\widehat{BC}$ be S', and the point of symmetry with respect to AC be S. Let the points of contact of circle ω_1 with circles O, O_1, O_2 be U, U_1, U_2, respectively. Let the projection of P on SS' be P'' and the projection of Q on SS' be Q''. From the congruency of the right triangles CO_2Q and $SP''P$, SP is perpendicular to CQ.

1. Orthocenter of triangle PQS.

Consider triangle PQS; its altitude from Q lies on CQ. From the congruency of triangle $PQ'''Q$ and triangle AOS, SA is perpendicular to PQ. From the congruency of triangles PO_1B and $QQ''S$, PB is perpendicular to QS. Therefore PB, CQ and SA meet in H, the orthocenter of triangle PQS.

2. The orthocenter of triangle PQS is on ω_1 and is the mid-point of $L\omega_1$.

Invert the figure with B the center of inversion, and BA BC the square of the radius of inversion.

$$BA \cdot BC = 2a \cdot 2c = 4ac.$$

Since A is transformed into C and the points on the circle of inversion are transformed into themselves, the circle O_1 inverts into itself with the sides interchanged. The circle O_2 is transformed into the perpendicular to AB at C, Cx. The circle O goes into the perpendicular to AB at A, Az. Hence the circle ω_1 inverts into a circle O_1' tangent to the circle O_1, to the line Cx and the line Az. Since the radius of O_1' is b, the point of tangency with circle O_1 must be P, and therefore P is the inverse of U_1. The circle O_1' is the same size as the circle O_1 and P is therefore the mid-point of the segment O_1O_1'. Since U_1 is the internal homothetic center of the circles ω_1 and O_1, the second point of intersection of BP with ω_1 is a point in which the tangent is parallel to BC. (Homothetic points are opposite ends of parallel radii.) Call this point R temporarily. The center, ω_1, is then the intersection of the line BO_1' with the line RL through ω_1 perpendicular to BC. Since P is the mid-point of O_1O_1', R is the mid-point of $L\omega_1$. Now invert the figure using C as the center of inversion and $|CB \cdot CA| = 4ab$ as the square of the radius of inversion. The circle O_2 is transformed into itself; the circle O_1 into By, the line perpendicular to BC at B; and the circle O into Az, the line perpendicular to BC at A. Therefore the circle ω_1 inverts into a circle tangent to the lines Az and By, and to the circle O_1. The radius of the inverted circle is c and therefore it must be tangent to O_2 at Q. Hence the circle O_2' is equal to the circle O_2, point ω_1 is on the line CO_2', and U_2 is the internal homothetic center of ω_1 and O_2. Therefore CQ cuts circle ω_1 in the point in which the tangent is again parallel to BC. But the lines BP and CQ meet in H, therefore R and H are the same point and H is on the circle ω_1 and is the mid-point of $\omega_1 L$. Hence H, the orthocenter of triangle PQS is on ω_1 and is the mid-point of $L\omega_1$.

3. Properties of trapezoid CMNB.

If the lines BP and CQ cut Cx and By in the points M and N respectively, the orthocenter H of triangle PQS is the point of intersection of the diagonals of the trapezoid CMNB. Hence H is the mid-point of the line through H parallel to the bases of the trapezoid CMNB. Since H is the mid-point of the line $L\omega_1$, the line MN passes through the center ω_1 of the circle ω_1.

4. Altitudes of triangles PSQ and points of tangency of circle ω_1 with the arbelos.

In the inversion with B as center, circles O_1 and O_1' were transformed into circles O_1 and ω_1 respectively, and P into U_1. Hence the altitude BP of triangle PQS passes through U_1, and by analogy SA passes through U and CQ through U_2.

5. Sets of concyclic points.

The following three sets of four points are each concyclic:

C, U_1, U_2, B

B, A, U_1, U

A, C, U, U_2.

Proof:

If a circle is tangent to two other circles, the line joining the points of contact will pass through a homothetic center of the two circles.* The circle ω_1 is tangent to the circles O_1 and O_2 at the points U_1 and U_2 respectively. Hence the line U_1U_2 passes through a homothetic center of the circles O_1 and O_2. Let this homothetic center be denoted by V. Then $VU_1 \cdot VU_2$ equals $VC \cdot CB$. This is a necessary and sufficient condition for the points U_1, U_2, C, and B to lie on a circle. In order to find the center, invert the figure with respect to B as center. The products $BC \cdot BA$ and $BP \cdot BU_1$ are equal since A, C, P and U_1 are concyclic. Then $BC \cdot BA = BP(BP - PU_1)$, from which we obtain

* Daus, College Geometry, p. 23.

$BP \cdot PU_1 = \overline{BP}^2 - BA \cdot BC.$

$$= \overline{BO_1}^2 + \overline{O_1P}^2 - (BO_1 + O_1C)(BO_1 - O_1A)$$

$$= \overline{BO_1}^2 + \overline{O_1P}^2 - \overline{BO_1}^2 - O_1C \cdot BO_1 + BO_1 \cdot O_1A + O_1C \cdot O_1A$$

$$= \overline{O_1P}^2 + BO_1(O_1A - O_1C) + \overline{O_1A}^2$$

$BP \cdot PU_1 = \overline{CP}^2$

Therefore the circle through BCU_1 is tangent to PC at C. The center of the circle is on the line perpendicular to PC at C and also on the perpendicular bisector of BC. Therefore the center is S.

Likewise B, A, U_P, U are on a circle whose center is Q and A, C, U, and U_2 are on a circle whose center is P.

6. Perpendicular bisectors of the sides of the triangle UU_1U_2.

The lines $S\omega_1$, $Q\omega_1$, and $P\omega_1$ are the perpendicular bisectors of the sides U_1U_2, U_1U and U_2U respectively.

Proof:

Since $SU_1 = SU_2$, S lies on the perpendicular bisector of U_1U_2;

since $QU = QU_1$, Q lies on the perpendicular bisector of UU_1; and

since $PU = PU_2$, P lies on the perpendicular bisector of UU_2.

7. The point ω_1 is on the circumcircle of triangle PQS.

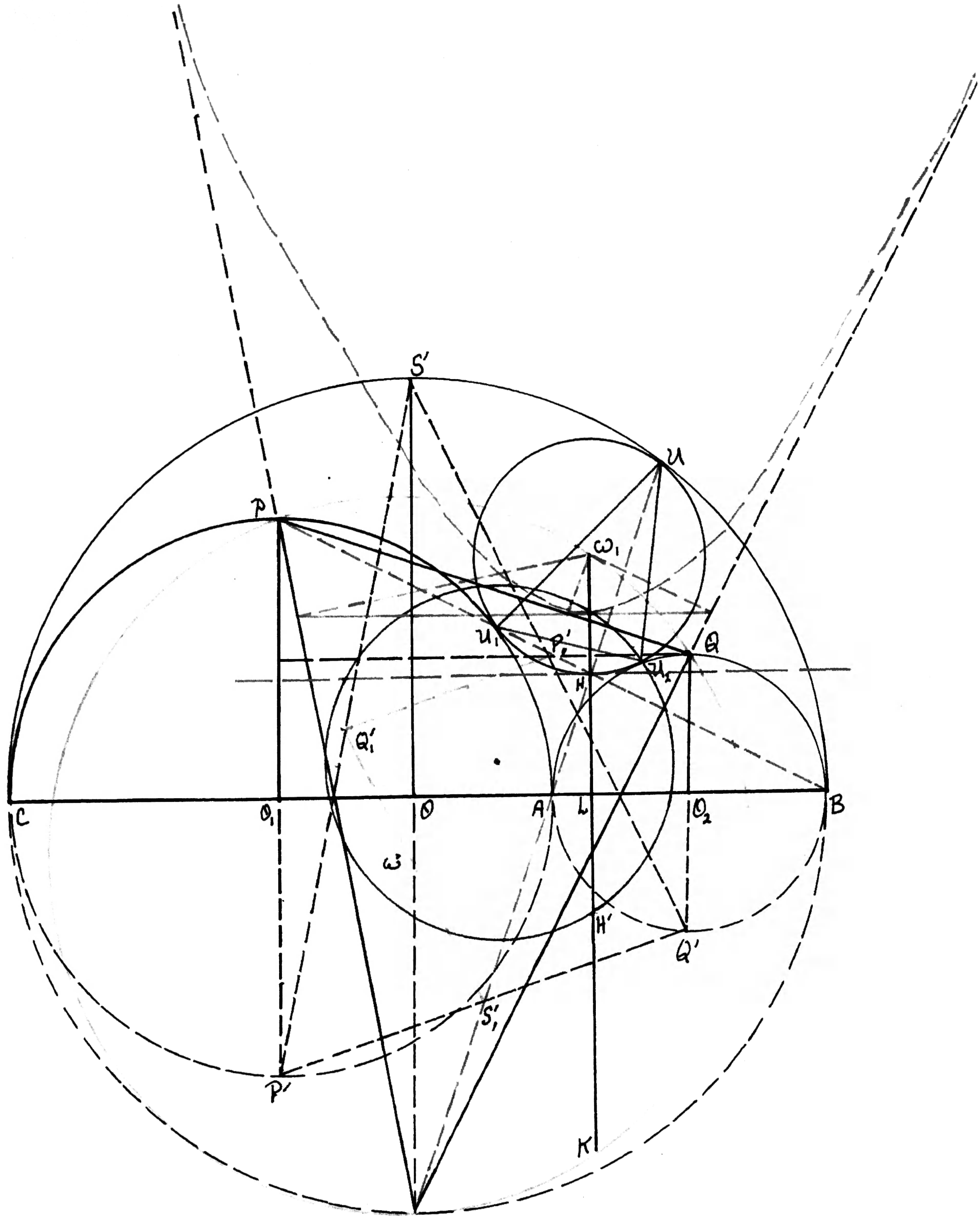

Figure 27

Proof:

$\angle BU_1U_2 = \angle BCU_2$, since they cut off the same arc on circle BCU_1U_2.

$\angle BCU_2 \doteq \angle BCQ$, since CQ is an extension of line CU_2.

$\angle QS\omega_1 = \angle BU_1U_2$ for QS is perpendicular to U_1B and $S\omega_1$ is perpendicular to U_1U_2.

$\angle S'SP = \angle BCQ$, since SS′ is perpendicular to BC and SP is perpendicular to CQ

$\therefore \angle QS\omega_1 = \angle S'SP$.

$\therefore$ SO and $S\omega_1$ are isogonal lines in triangle PQS.

Similarly the lines PO_1 and $P\omega_1$ are isogonal conjugates and also the lines QO_2 and $Q\omega_1$.

Since the lines SO, PO_1 and QO_2 are parallel they meet in a point at infinity, and the isogonal conjugate of the point at infinity is on the circumcircle of triangle PQS.* Hence the point ω_1 lies on the circumcircle.

8. Simson line. (Figure 27)

From ω_1, a point on the circumcircle of the triangle PQS, drop perpendiculars to the sides of the triangle. The line joining the feet of the perpendiculars is the Simson line of ω_1 with respect to the triangle PQS. The Simson line of a point on the circumcircle is perpendicular to the isogonal conjugate of the line joining the

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* Daus, College Geometry, p. 70.

point to a vertex.* Therefore, the Simson line of ω_1 with respect to the triangle PQS is perpendicular to the lines SO, PO_1, and QO_2 and is therefore parallel to the line BC.

9. Nine-point circle.

The Simson line of a point on the circumcircle bisects the line joining the point to the orthocenter and the point of bisection lies on the nine-point circle.** Therefore the Simson line of ω_1 with respect to the triangle PQS bisects $\omega_1 H$ in a point on the nine-point circle of the triangle PQS.

10. Inscribed parabola.

The orthocenter of a triangle formed by any three tangents to a parabola is a point on the directrix. Also, the circle circumscribed about the triangle formed by three tangents to a parabola passes through the focus; for the feet of the perpendiculars from the focus to these tangents are collinear.*** Then ω_1 is the focus, the point of intersection of the Simson line of ω_1 and the nine-point circle is the vertex and a line parallel to the Simson line, passing through H is the directrix of a parabola which is tangent to each of the three sides of the triangle PQS.

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* Daus, College Geometry, p. 138.

** Ibid., p. 137.

*** Casey, Analytical Geometry, p. 178.

11. FORMULA RELATING THE RADII OF THE THREE CIRCLES OF THE ARBELOS, THAT OF THE INSCRIBED CIRCLE AND THAT OF THE CIRCUMCIRCLE OF TRIANGLE PQS.

Let the mid-points of the sides of the triangle $P'Q'S'$ be denoted by P_1', Q_1', S_1' where P_1' lies on $Q'S'$ etc. By the construction $SP'AQ'$ is a rectangle; its diagonals bisect each other, hence S_1' lies on SA. Likewise $PS'BQ'$ is a rectangle by construction; its diagonals bisect each other and therefore P_1' lies on BP.

$\angle PHS + \angle PQS = 180°$, for PB and SA are altitudes of triangle PQS

$\therefore \angle PHS = 180° - \angle PQS$

Hence $\angle P_1'HS_1' = 180° - \angle PQS$, for P_1' and S_1' lie on PH and SH respectively.

$\angle PQS = \angle P'Q'S'$ by symmetry.

$\therefore \angle P_1'HS_1' = 180° - \angle S'Q'P'$.

$\angle S'Q'P' = \angle S_1'Q_1'P_1'$ for $S_1'Q_1'$ is parallel to $S'Q'$ and $Q_1'P_1'$ is parallel to $P'Q'$

$\therefore \angle P_1'HS_1' = 180° - \angle S_1'Q_1'P_1'$ and P_1', H, S_1', Q_1' are concyclic.*

Hence H lies on the nine-point circle of triangle $P'Q'S'$. By symmetry the orthocenter H' of triangle $S'Q'P'$ will lie on the nine-point circle of triangle PQS.

Let the line $\omega_1 H$ cut circle ω, the circumcircle of triangle PQS in a point K.

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* Daus, College Geometry, p. 5.

Then $HK = 2HH'$, for the nine-point circle of a triangle bisects any line from the orthocenter to the circumcircle.*

$= 2\ (2 \cdot HL)$

$= 4\ H\omega_1$

$|HK \cdot H\omega_1| = 4\ \overline{H\omega_1}^2 = 4\rho_1^2$, where ρ_1 denotes the radius of circle ω_1, but $|HK \cdot H\omega_1| = |\overline{H\omega}^2 - \rho^2|$, where ρ denotes the radius of the circumcircle of the triangle PQS.

The distance from the orthocenter to the circumcenter of a triangle is given by the formula

$$\overline{OH}^2 = 9R^2 - (a_1^2 + a_2^2 + a_3^2)^{**}$$

where O is the circumcenter, H is the orthocenter, R is the radius of the circumcircle, and a_1, a_2, a_3 are the sides of the triangle.

$$\therefore\ \overline{H\omega}^2 = 9\rho^2 - (\overline{PQ}^2 + \overline{QS}^2 + \overline{SP}^2).$$

But $PQ = SA$ from $\triangle S\hat{O}A$ and $\triangle PQO'''$

$$\therefore\ \overline{PQ}^2 = \overline{SA}^2 = a^2 + (a-2c)^2$$

$$= 2a^2 - 4ac + 4c^2$$

$$= 2(b+c)^2 - 4c(b+c) + 4c^2$$

$$= 2(b^2 + 2bc + c^2) - 4bc - 4c^2 + 4c^2$$

* Winsor, Modern Higher Plane Geometry, p. 73.

** Johnson, Modern Geometry, p. 175.

$$= 2(b^2+c^2)$$

$$\overline{QS}^2 = (a+c)^2 + (a-c)^2$$

$$= 2(a^2+c^2)$$

$$\overline{SP}^2 = (a+b)^2 + (a-b)^2$$

$$= 2(a^2+b^2)$$

Then $\overline{H\omega}^2 = 9\rho^2 - 2(b^2+c^2) - 2(a^2+c^2) - 2(a^2+b^2)$

$$= 9\rho^2 - 4(a^2+b^2+c^2)$$

But $4\rho_1^2 = |\overline{H\omega}^2 - \rho^2|$

$\therefore \quad 4\rho_1^2 = |9\rho^2 - 4(a^2+b^2+c^2) - \rho^2|$

$$4\rho_1^2 = 8\rho - 4a^2 - 4b^2 - 4c^2$$

$$\rho_1^2 + a^2 + b^2 + c^2 = 2\rho^2 .$$

12. Relation between the triangle consisting of the points of tangency of the circle inscribed in the arbelos and the triangle PQS.

$\angle U_1HU_2$ = $\angle$PHQ, because of extended lines.

= $\angle$CHB, because of vertical angles.

= 180° - $\angle$PSQ, for SP is perpendicular to CQ and BP is perpendicular to SQ.

$\angle UHU_1$ = $\angle$PQS, since they are supplements of $\angle$UHB

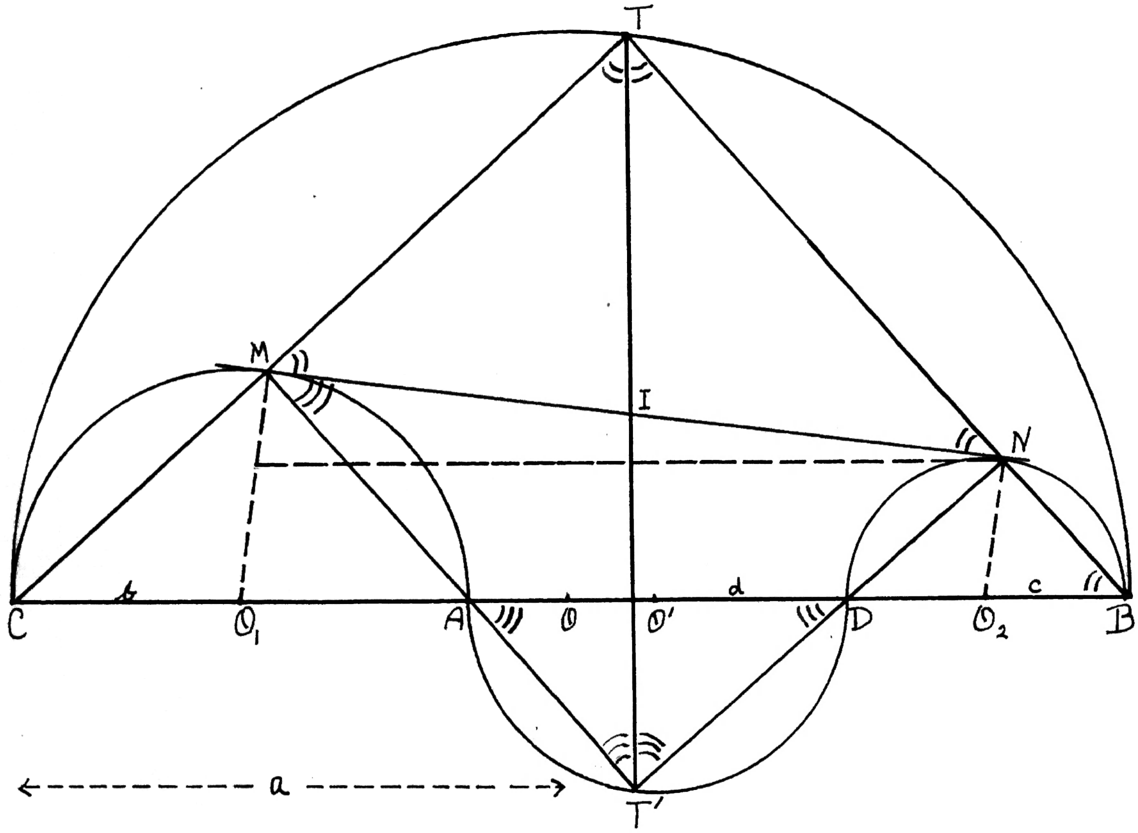

Figure 28

$\angle U_2HU = \angle QPS$, since they are supplements of $\angle UHC$.

$$\angle U_1HU_2 + \angle HU_1U + \angle U_1UU_2 + \angle UU_2H = 360°$$

$$\begin{aligned}\angle U_1UU_2 &= 360° - \angle PHQ - (\angle HU_1U + \angle UU_2H)\\ &= 360° - \angle PHQ - 180°\\ &= 180° - \angle PHQ\\ &= \angle PSQ\end{aligned}$$

$\angle UU_2U_1 = \angle UHU_1$, as they cut off the same arc on circle ω_1

$= \angle PQS$

$\angle U_2U_1U = \angle U_2HU$, since both cut the same arc on circle ω_1

$= \angle QPS$

Since their angles have been shown to be equal, the triangles PQS and UU_1U_2 are similar.

The line UU_1 meets PS in the internal homothetic center of circles O_1 and O, line UU_2 cuts QS in the internal homothetic center of circles O_2 and O, and line U_1U_2 intersects PQ in the external homothetic center of circles O_1 and O_2.

H. <u>ARBELOS FORMED BY FOUR SEMI-CIRCLES</u>. (Figure 28)

<u>On the diameter BC of a circle O, let two points, A and D, be marked between B and C, and let the half-circles O_1 and O_2 on AC and DB as diameters be drawn above BC and the half-circle O' on DA as a diameter be drawn below BC. Let the radical axis of the circles O_1 and O_2 cut the</u>

circle O in T and the circle O' in T'. Then the lines CT and AT' meet on the circle O_1 and the lines DT' and BT meet on the circle O_2 in the points of contact, M and N, of the common external tangent of the circles O_1 and O_2; the figure $MTNT'$ is a rectangle and the area bounded by the semi-circumferences O, O_1, O_2 above BC and O' below BC is equal to the area of the circle with TT' as diameter.

Proof:

Since the point T is on the radical axis,

$$CT \cdot MT = BT \cdot NT,$$

$$\text{or } \frac{CT}{BT} = \frac{NT}{MT}.$$

Then the triangle CTB is similar to the triangle NTM.

Therefore $\angle C = \angle N$

and $\angle B = \angle M$,

but angle C equals angle $T'TB$ since both angles are complements of angle CTT', and therefore angle N equals angle $T'TB$. Also angle B equals angle $T'TC$, since both are complements of angle BTT' and hence angle M equals angle $T'TC$. Then if the point where MN intersects $T'T$ is denoted by I, the triangles MIT and NIT are isosceles because each has two equal angles. Therefore MI equals IN since both equal IT.

Since the point C is the external homothetic center of circles O_1 and O, O_1M and OT are parallel. Likewise, since B is the external homothetic center of circles O_2 and O, O_2N and OT are parallel. Hence

$$O_1M \parallel O_2N.$$

Since I is on the radical axis and IM equals IN and since M and N are the ends of parallel radii, the line MN is the common external tangent of the circles O_1 and O_2.

Draw AT' and DT' and call their point of intersection with O_1 and O_2, M' and N' respectively. Then

$$T'A \cdot T'M' = T'D \cdot T'N'.$$

or $$\frac{T'A}{T'D} = \frac{T'N'}{T'M'},$$ since T' is on the radical axis of circle O_1 and O_2.

Therefore the triangles $T'AD$ and $T'N'M'$ are similar and hence angle A equals angle N' and angle D equals angle M'. But angle A equals angle $TT'D$, since both are complements of angle $AT'T$, and angle D equals angle $TT'A$ since both are complements of angle $TT'D$. Hence angle N' equals angle $TT'D$ and angle M' equals angle $TT'A$ and if the point of intersection of $M'N'$ and TT' is called I', the triangles $M'I'T'$ and $T'I'N'$ are isosceles. Hence $M'T' = I'T' = I'N'$ or $M'I' = I'N'$.

Since A is the internal homothetic center of circles O' and O_1, the lines O_1M' and $O'T'$ are parallel. In the same way, since D is the internal homothetic center of circles O_2 and O', the lines O_2N' and $O'T'$ are parallel. Hence

$$O_1M' \parallel O_2N'.$$

Since $I'M' = I'N'$ and I' is on the radical axis of circles O_1 and O_2 and since M' and N' are the ends of parallel radii,

$M'N'$ is the common external tangent of circles O_1 and O_2. M' therefore coincides with M, N' with N and I' with I and hence the lines CT and AT$'$ have been shown to meet in M and the lines BT and DT$'$ in N.

Since the angles AMC, CTB, DNB and AT$'$D are each inscribed in a semi-circle, each angle is a right angle and therefore MTNT$'$ is a rectangle.

The area of CTBNDT$'$AM $= 4\pi\left(\frac{TT'}{2}\right)^2 = \pi \overline{TT'}^2$.

Proof:

$$\overline{MN}^2 = \overline{TT'}^2 = \overline{O_1O_2}^2 - (O_1M - O_2N)^2 = (2a-b-c)^2 - (b-c)^2$$
$$= [(2a-b-c)-(b-c)][(2a-b-c)+(b-c)]$$
$$= (2a-2b)(2a-2c) = AB \cdot CD$$
$$= 4(a-b)(a-c)$$

The area of the circle on TT$'$ as diameter

$$= \pi\left(\frac{TT'}{2}\right)^2$$
$$= \pi/4\,[4(a-b)(a-c)]$$
$$= \pi(a-b)(a-c).$$

The area S, within the circle O, bounded by the semi-circumferences O, O_1, O', O_2 has the following value:

$$S = \frac{\pi}{2}\left[a^2-b^2-c^2+\left(\frac{2a-2b-2c}{2}\right)^2\right]$$
$$= \frac{\pi}{2}\left(a^2-b^2-c^2+a^2+b^2+c^2-2ab-2ac+2bc\right)$$
$$= \pi/2\,[a(a-b)-c(a-b)]$$

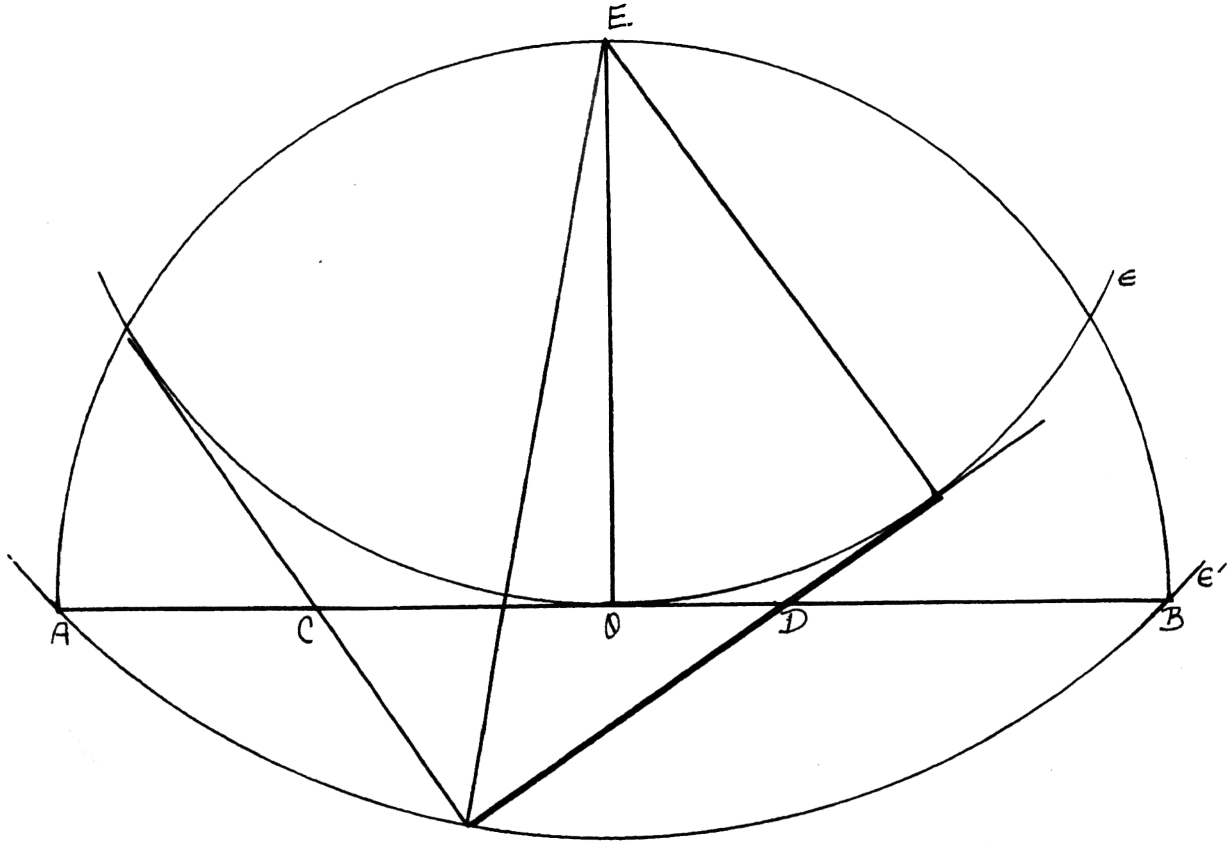

Figure 29

$$= \pi(a-b)(a-c).$$

When the semi-circles O_1 and O_2 are tangent at A, the area S, thus found, is that of the arbelos. When the semi-circles O_1 and O_2 have equal radii, the figure bounded by the four semi-circles was considered by Archimedes in the "Book of Lemmas" and is called the "Salinon" or "Feuille d'ache."

If $d^2 = b^2 + c^2$, the curve CMAT′DNB separates the area of the circle O into two arbeloi having equal areas and equal perimeters.

The quantity d^2 will equal $b^2 + c^2$ when $\overline{CD}^2 = \overline{AC}^2 + \overline{DB}^2$ or CD, AC, and DB must be the hypotenuse and the two legs of a right triangle.

Consideration of this triangle suggests an easy construction of the two points C and D so that they will fulfill the preceding condition. (Figure 29) Draw the radius OE of the semi-circle perpendicular to AB, and draw the circumferences $\mathcal{E} \equiv (E, OA)$, $\mathcal{E}' \equiv (E, EA = OA\sqrt{2})$. The tangents drawn to the circumference $\mathcal{E}$ from any point on the arc $\mathcal{E}'$ in the lower half of the plane determine on AB the desired two points C and D.

If the semi-circle O is given, the relations $b + c + d = a$ and $d^2 = b^2 + c^2$ enable one to calculate the radii of two of the semi-circles O_1, O_2, O_3 when the third is known.

If b is given,

$$d + c = a - b \qquad\qquad d^2 - c^2 = b^2$$

$$d - c = \frac{b^2}{a-b} \qquad\qquad d - c = \frac{b^2}{d+c}$$

$$2d = \frac{(a-b)^2 + b^2}{(a-b)} \qquad\qquad d - c = \frac{b^2}{a-b}$$

$$d = \frac{(a-b)^2 + b^2}{2(a-b)}$$

$$2c = \frac{(a-b)^2 - b^2}{a-b}$$

$$= \frac{a^2 - 2ab}{a-b}$$

$$c = \frac{a(a-2b)}{2(a-b}.$$

By analogy, if c is given

$$b = \frac{a(a-2c)}{2(a-c)} \qquad\qquad d = \frac{(a-c)^2 + c^2}{2(a-c)}$$

Finally, if d is given

$$b + c = a - d \quad \text{and} \quad b^2 + c^2 = d^2$$

$$b^2 + 2bc + c^2 = a^2 - 2ad + d^2$$

$$b^2 + c^2 = d^2$$

$$2bc = a^2 - 2ad$$

$$bc = a/2\,(a - 2d).$$

Consequently b and c are the two roots of the equation

$$2x^2 - 2(a - d)x + a(a - 2d) = 0$$

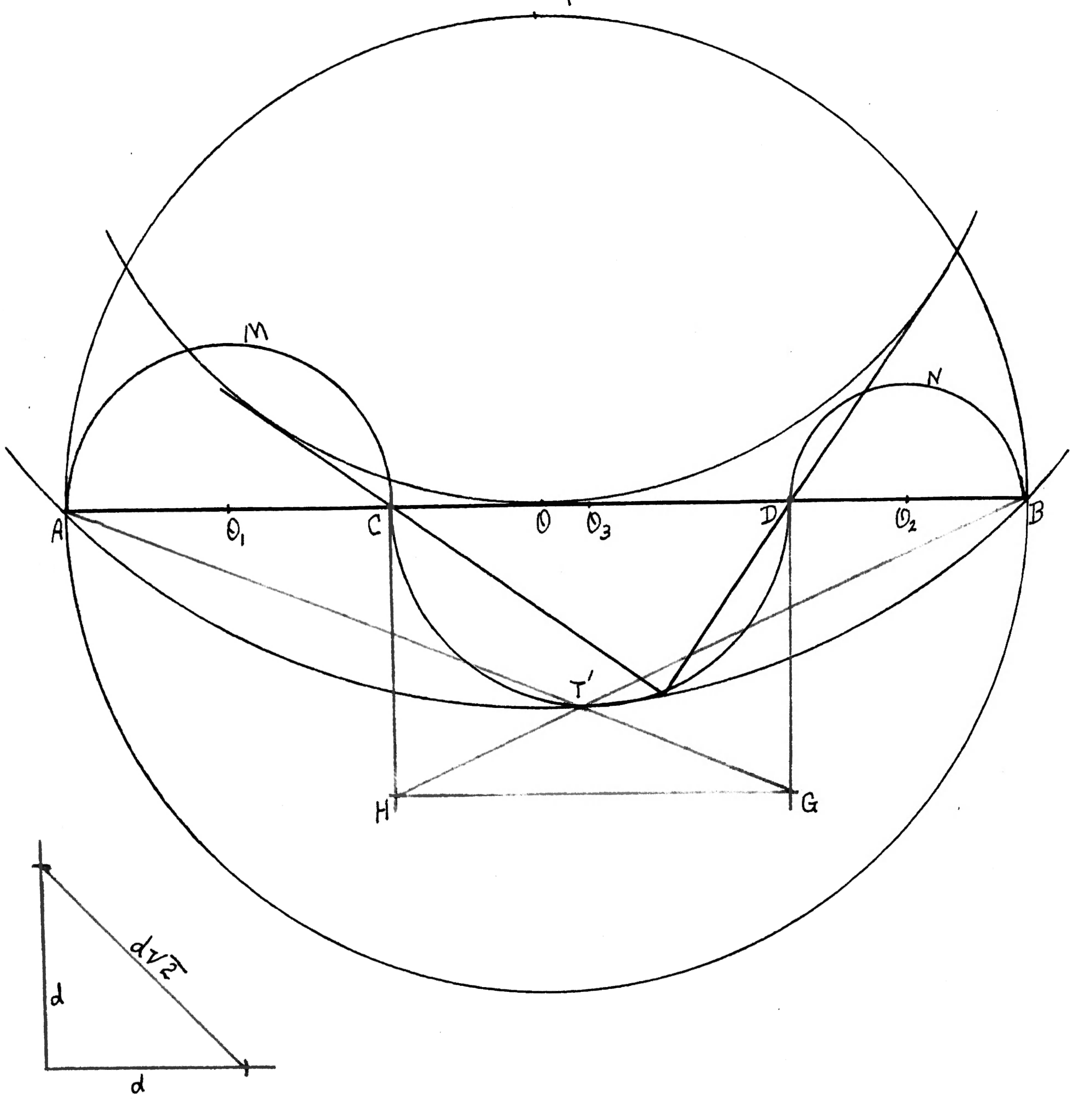

Figure 30

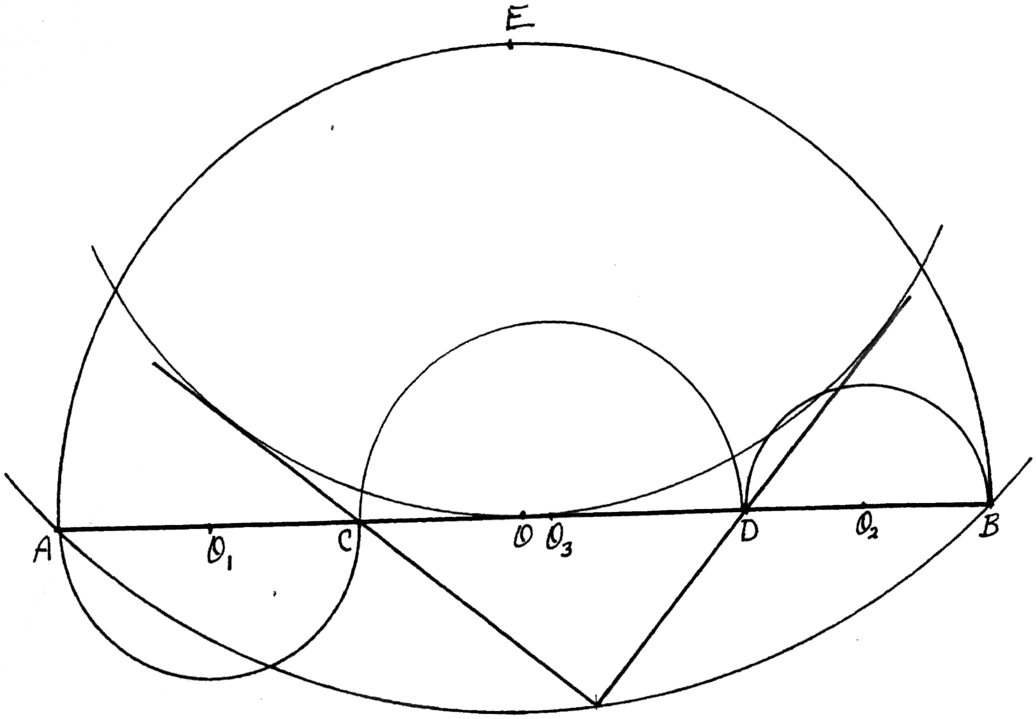

Figure 31

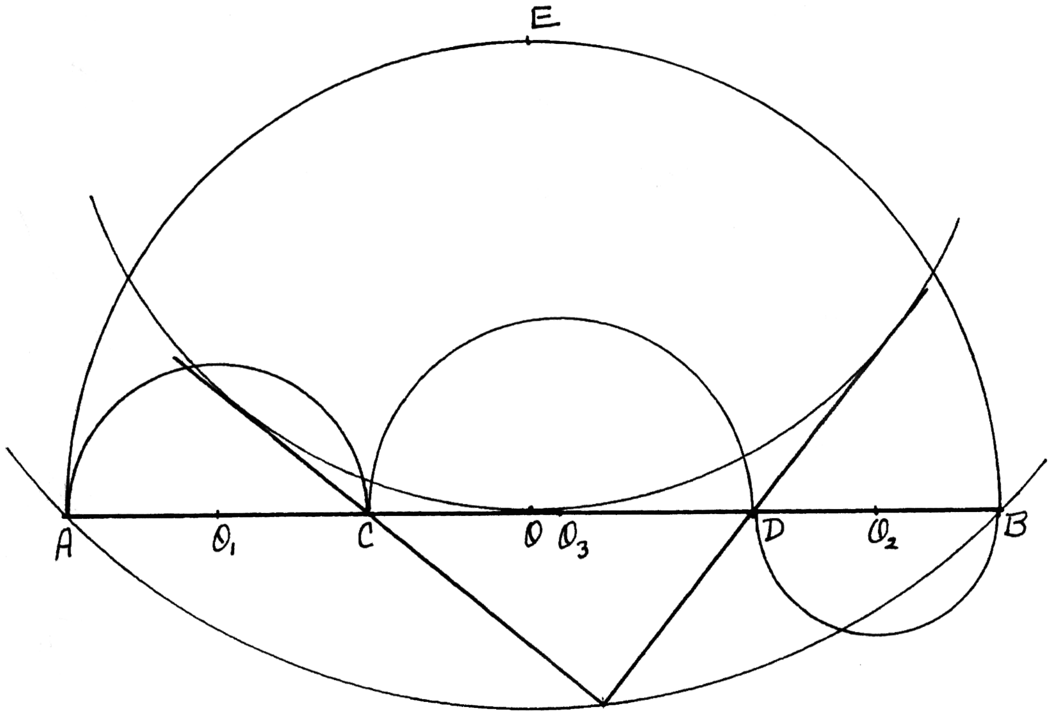

Figure 32

Then the values of the roots are

$$x = \frac{2(a-d) \pm \sqrt{4(a-d)^2 - 8a(a-2d)}}{4}$$

$$= \frac{a-d \pm \sqrt{a^2 - 2ad + d^2 - 2a^2 + 4ad}}{2}$$

$$= \frac{a-d \pm \sqrt{(a+d)^2 - 2a^2}}{2}$$

It follows from the theorem of Fermat and its reciprocal* that if the semi-circle O_3 is enclosed in a rectangle CDGH of height CH = $d\sqrt{2}$ equal to a side of the square inscribed in the circle O_3, the lines AG and BH will meet on the circle. (Figure 30)

If the semi-circles O_1 or O_2 and O_3 are replaced by the semi-circles symmetric to them with respect to the line AB, thus interchanging d and b or d and c, the area bounded by the four corresponding semi-circles has the value: (Figures 31 and 32)

$$S_1 = \frac{\pi}{2}(a^2 + b^2 - c^2 - d^2) = \pi(a-c)(a-d) = \pi(b+c)(b+d)$$

$$\text{or } S_2 = \frac{\pi}{2}(a^2 - b^2 + c^2 - d^2) = \pi(a-b)(a-d) = \pi(b+c)(c+d).$$

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* V. Thebault gives "Nova opera mathematica," 1680, p. 118.

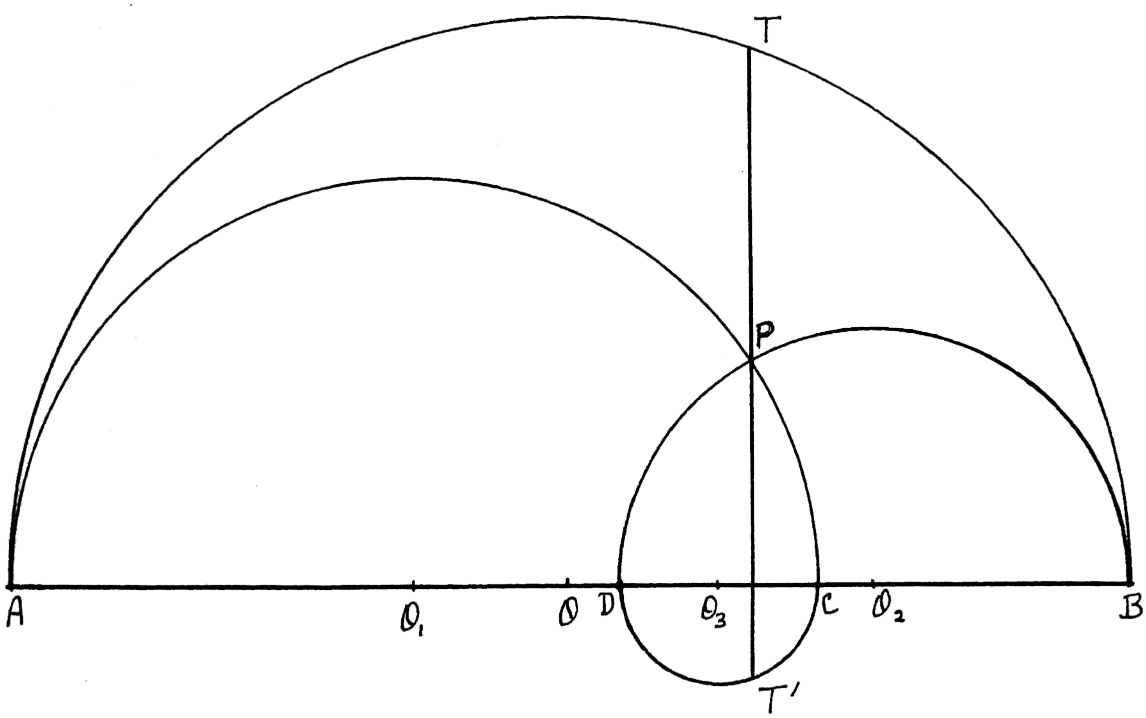

Figure 33

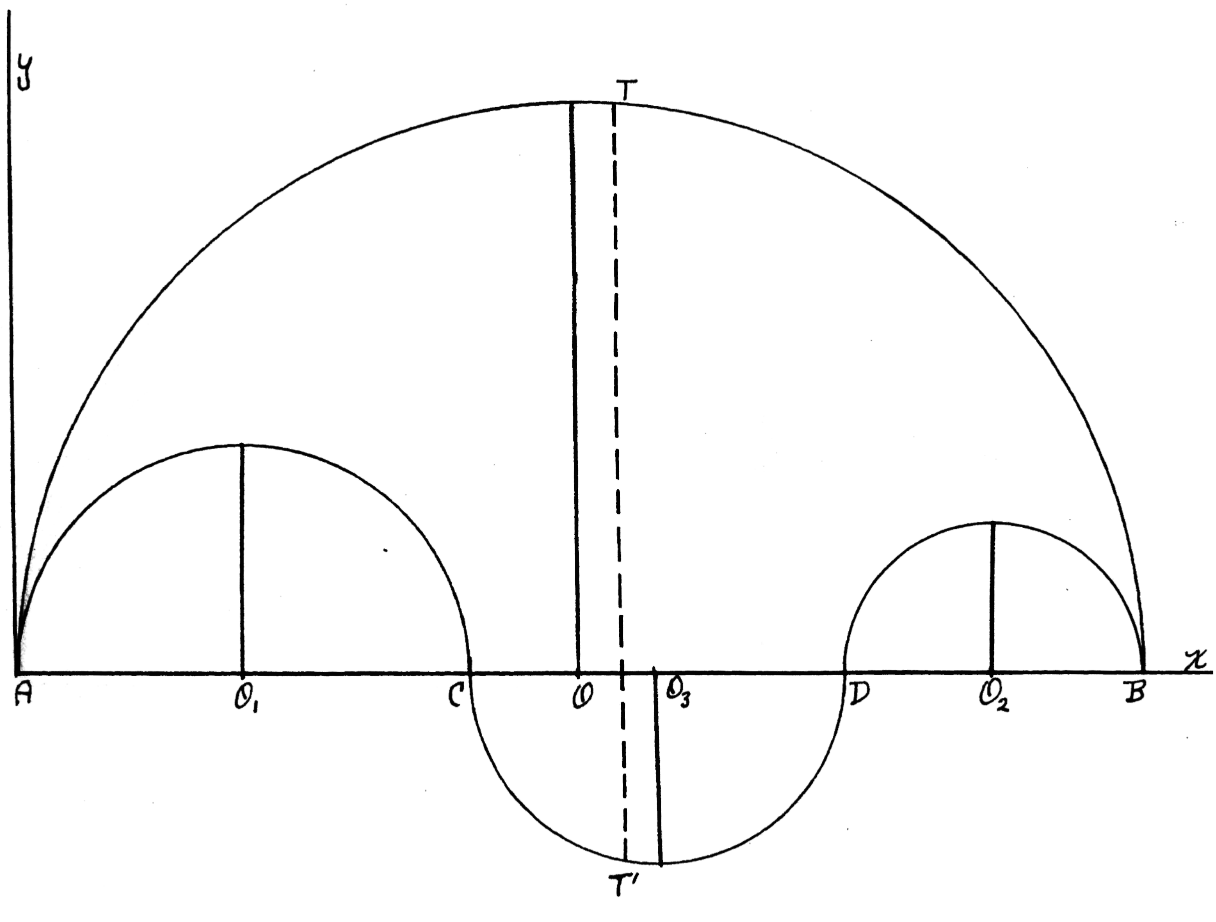

Figure 34

We have assumed before that the circles O_1 and O_2 were exterior to each other. The properties of **Article** H hold if they cut each other. (Figure 33) If P is the point of intersection, the arbelos is formed by the parts of the plane bounded, the first by the semi-circumference O and the arcs **AP** and BP of the semi-circumferences O_1 and O_2, the second by the semi-circumference O_3 and the arcs CP and DP of the semi-circumferences O_1 and O_2. The circles inscribed in the mixtilinear triangles APT and BPT have been proved equal in **Article** C, 2, of section II.

I. CENTROIDS (Figure 34).

Let the figure have the axes **Ax** and **Ay** as **drawn.** Let g, g_1, g_2, g_3 be the centers of gravity of the semi-circumferences O, O_1, O_2, O_3 and let G, G_1, G_2, G_3 be the centers of gravity of the corresponding areas. These points are on lines perpendicular to AB at O, O_1, O_2, O_3 and a distance from this line easily calculated by the theorem of Pappus.

For the arcs:

$$g\left(a, \frac{2a}{\pi}\right), \quad g_1\left(b, \frac{2b}{\pi}\right), \quad g_2\left(2b+2d+c, \frac{2c}{\pi}\right), \quad g_3\left(2b+d, -\frac{2d}{\pi}\right)$$

and for the areas:

$$G\left(a, \frac{4a}{3\pi}\right), \quad G_1\left(b, \frac{4b}{3\pi}\right), \quad G_2\left(2b+2d+c, \frac{4c}{3\pi}\right), \quad G_3\left(2b+d, \frac{-4d}{3\pi}\right).$$

The center of gravity of the arc ATBDT′CA bounding the arbelos is then

for $\overline{x}$

$$(\pi a+\pi b+\pi c+\pi d)\,\overline{x} = \pi a(a)+\pi b(b)+\pi c\,(2b+2d+c)+\pi d(2b+d)$$

$$(a+b+c+d)\,\overline{x} = a^2+b^2+2bc+2cd+c^2+2bd+d^2$$

$$\overline{x} = \frac{a^2+b^2+2bc+2cd+c^2+2bd+d^2}{a+b+c+d}$$

$$= \frac{a^2+b^2+c^2+d^2+2bc+2cd+2bd}{2a}$$

$$= \frac{a^2+(b+c+d)^2}{2a}$$

$$= \frac{a^2+a^2}{2a}$$

$$= \frac{2a^2}{2a}$$

$$= a$$

for $\overline{y}$

$$(\pi a+\pi b+\pi c+\pi d)\,\overline{y} = \pi a\left(\tfrac{2a}{\pi}\right)+\pi b\left(\tfrac{2b}{\pi}\right)+\pi c\left(\tfrac{2c}{\pi}\right)-\pi d\left(\tfrac{2d}{\pi}\right)$$

$$\pi(a+b+c+d)\,\overline{y} = 2(a^2+b^2+c^2-d^2)$$

$$\overline{y} = \frac{2(a^2+b^2+c^2-d^2)}{\pi(a+b+c+d)}$$

but $a = b+c+d$

$$a^2+b^2+c^2-d^2 = a^2+b^2+c^2-(a-b-c)^2$$

$$= a^2 + b^2 + c^2 - (a^2 + b^2 + c^2 - 2ab - 2ac + 2bc)$$

$$= 2ab + 2ac - 2bc$$

$$\text{and} \quad b + c = a - d$$

$$\text{Then} \quad \bar{y} = \frac{4(ab + ac - bc)}{2\pi a}$$

$$= \frac{2[a(b+c) - bc]}{\pi a}$$

$$= \frac{2[a(a-d) - bc]}{\pi a}$$

$$\therefore \quad \bar{y} = \frac{2(a^2 - ad - bc)}{\pi a}$$

The center of gravity of the area of ATBDT′CA of the arbelos is then

for $\overline{X}$,

$$\left(\frac{\pi a^2}{2} - \frac{\pi b^2}{2} - \frac{\pi c^2}{2} + \frac{\pi d^2}{2}\right)\overline{X} = \frac{\pi a^3}{2} - \frac{\pi b^3}{2} - \frac{\pi c^2}{2}(2b + 2d + c) + \frac{\pi d^2}{2}(2b + d)$$

$$(a^2 - b^2 - c^2 + d^2)\overline{X} = a^3 - b^3 - c^3 + d^3 - 2bc^2 - 2dc^2 + 2bd^2$$

$$\therefore \quad \overline{X} = \frac{a^3 - b^3 - c^3 + d^3 - 2(bc^2 + dc^2 - bd^2)}{2(b+d)(c+d)}$$

for $\overline{Y}$,

$$\frac{\pi}{2}(a^2 - b^2 - c^2 - d^2)\overline{Y} = \frac{\pi a^2}{2}\left(\frac{4a}{3\pi}\right) - \frac{\pi b^2}{2}\left(\frac{4b}{3\pi}\right) - \frac{\pi c^2}{2}\left(\frac{4c}{3\pi}\right) + \frac{\pi d^2}{2}\left(\frac{-4d}{3\pi}\right)$$

$$(a^2 - b^2 - c^2 - d^2)\overline{Y} = \frac{4}{3\pi}(a^3 - b^3 - c^3 - d^3)$$

$$\bar{Y} = \frac{4(a^3 - b^3 - c^3 - d^3)}{3\pi(b+d)(c+d)2}$$

$$= \frac{2(a^3 - b^3 - c^3 - d^3)}{3\pi(b+d)(c+d)}$$

but $a = b + c + d$

and $a^3 = (b+c+d)^3$

$$= b^3 + c^3 + d^3 + 3cb^2 + 3cd^2 + 3bd^2$$

$$+ 3bc^2 + 3b^2d + 3c^2d + 6bcd$$

$$\bar{Y} = \frac{2 \cdot 3(cb^2 + cd^2 + bd^2 + bc^2 + b^2d + c^2d + 2bcd)}{3\pi(b+d)(c+d)}$$

$$= \frac{2[c(b^2+2bd+d^2) + bd(b+d) + c^2(b+d)]}{\pi(b+d)(c+d)}$$

$$= \frac{2(b+d)(c^2+bd+cb+cd)}{\pi(b+d)(c+d)}$$

$$= \frac{2(b+d)(c+d)(c+b)}{\pi(b+d)(c+d)}$$

$$\therefore \bar{Y} = \frac{2(c+b)}{\pi}$$

The positions of these centroids hold true regardless of the positions of the points C and D.

When d = 0, the points C and D coincide and the preceding formulas reduce to those found for the knife of

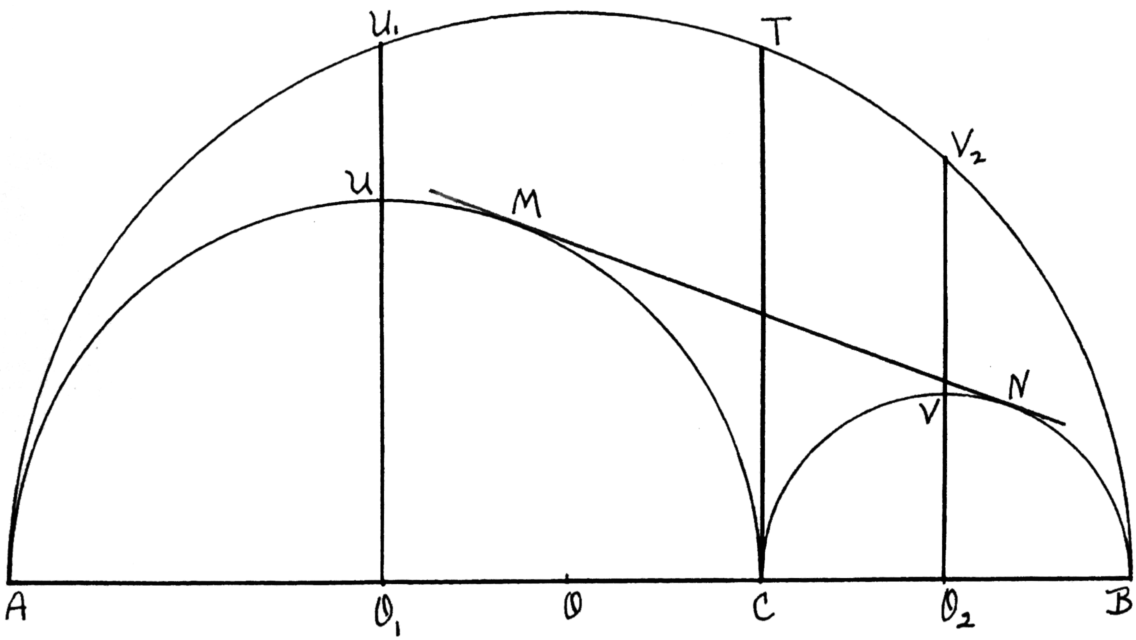

Figure 35

Archimedes; $x = a$, $y = \frac{2(a^2 - bc)}{\pi a}$ and $\overline{X} = \frac{1}{2}(3b + c)$, $\overline{Y} = \frac{2a}{\pi}$.

These properties may easily be extended to the case in which AB is divided into more than three semi-circles, the ones interior and the others exterior to the semi-circle O.

J. <u>GENERATED</u> <u>SURFACES</u>. (Figure 35)

Let U, V, and T be the points of intersection of the perpendiculars to AB at O_1, O_2, and C with the semi-circumferences O_1, O_2, and O, and let U_1, V_2 be the points in which the lines O_1U and O_2V cut the semi-circle O.

When the figure revolves about AB, the areas described by the rectilinear segments UU_1 and VV_2 have equal values for:

$$\pi(\overline{O_1U_1}^2 - \overline{O_1U}^2) = \pi[b(b + 2c) - b^2] = 2\pi bc,$$

and $$\pi(\overline{O_2V_2}^2 - \overline{O_2V}^2) = \pi[c(c + 2b) - c^2] = 2\pi bc.$$

Further, each one of these is double the area of the knife and also double the area of the circle of diameter equal to CT.

The surfaces of the zones described by the arcs U_1T and TV_2 of the semi-circumference O in turning about AB are respectively equal to:

$$2\pi a \cdot O_1C = 2\pi ab \quad \text{and} \quad 2\pi a \cdot CO_2 = 2\pi ac;$$

their sum is $2\pi a(b+c) = 4\pi a^2,$

which is eight times the area of the semi-circle O.

If the mixtilinear trapezium CUU_1T is revolved about AB, the differences of the areas generated by its curved sides TU_1 and CU is:

$$2\pi ab - 2\pi b^2 = 2\pi b(a-b) = 2\pi bc$$

and the differences of the areas generated by the bases CT and UU_1 is:

$$\pi\overline{CT}^2 - \pi\overline{UU_1}^2 = 4\pi bc - 2\pi bc = 2\pi bc.$$

These two differences are then equal and their common value is twice the area of the knife. These properties are applicable to the mixtilinear trapezium CVV_2T.

The lateral area of the truncated cone generated by the common external tangent MN of the semi-circles O_1, O_2 is equal to: $\pi\overline{MN}^2 = 4\pi bc$. It is four times the area of the knife.

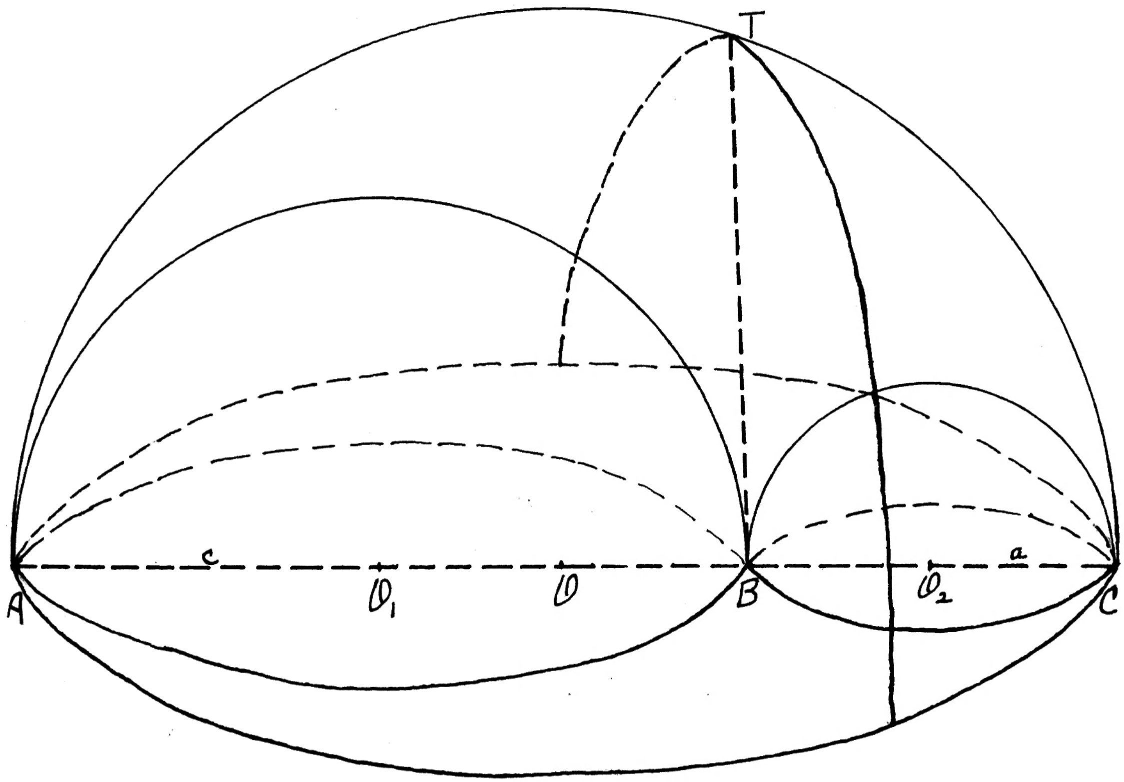

Figure 36

CHAPTER IV

SOLID AND HARMONIC ARBELOI

A. THE SOLID FORMED BY REVOLVING THE ARBELOS THROUGH 180° ABOUT ITS DIAMETER. (Figure 36)

1. Surface.

The surface of the solid formed by revolving the arbelos through 180° about its diameter AC is equal to the surface of the total sphere on AC as diameter.

Proof:

$$S = 2\pi \overline{AC} + 2\pi \overline{AB} + 2\pi \overline{BC}$$

$$= 2\pi(\overline{AC} + \overline{AB} + \overline{BC})$$

$$= 4\pi \overline{AC}$$

$=$ surface of the total sphere on $\overline{AC}$.

2. Volume.

$$V = \tfrac{1}{2}\left(\tfrac{4}{3}\pi \frac{\overline{AC}^3}{8} - \tfrac{4}{3}\pi \frac{\overline{AB}^3}{8} - \tfrac{4}{3}\pi \frac{\overline{BC}^3}{8}\right)$$

$$= \tfrac{1}{2}\left[\tfrac{4\pi}{24}(\overline{AC}^3 - \overline{AB}^3 - \overline{BC}^3)\right]$$

$$= \tfrac{1}{2}\,\tfrac{\pi}{6}\left[(\overline{AB} + \overline{BC})^3 - \overline{AB}^3 - \overline{BC}^3\right]$$

$$= \tfrac{\pi}{12}(\overline{AB}^3 + 3\overline{AB}^2\cdot\overline{BC} + 3\overline{AB}\cdot\overline{BC}^2 + \overline{BC}^3 - \overline{AB}^3 - \overline{BC}^3)$$

$$= \tfrac{\pi}{4}(\overline{AB}^2\cdot\overline{BC} + \overline{AB}\cdot\overline{BC}^2)$$

$$= \tfrac{\pi}{4}(\overline{AB}\cdot\overline{BC})(\overline{AB} + \overline{BC})$$

$$= \tfrac{\pi}{4}\,\overline{AB}\cdot\overline{BC}\cdot\overline{AC} \quad \text{or} \quad \tfrac{\pi}{4}\,\overline{TB}^2\cdot\overline{AC}$$

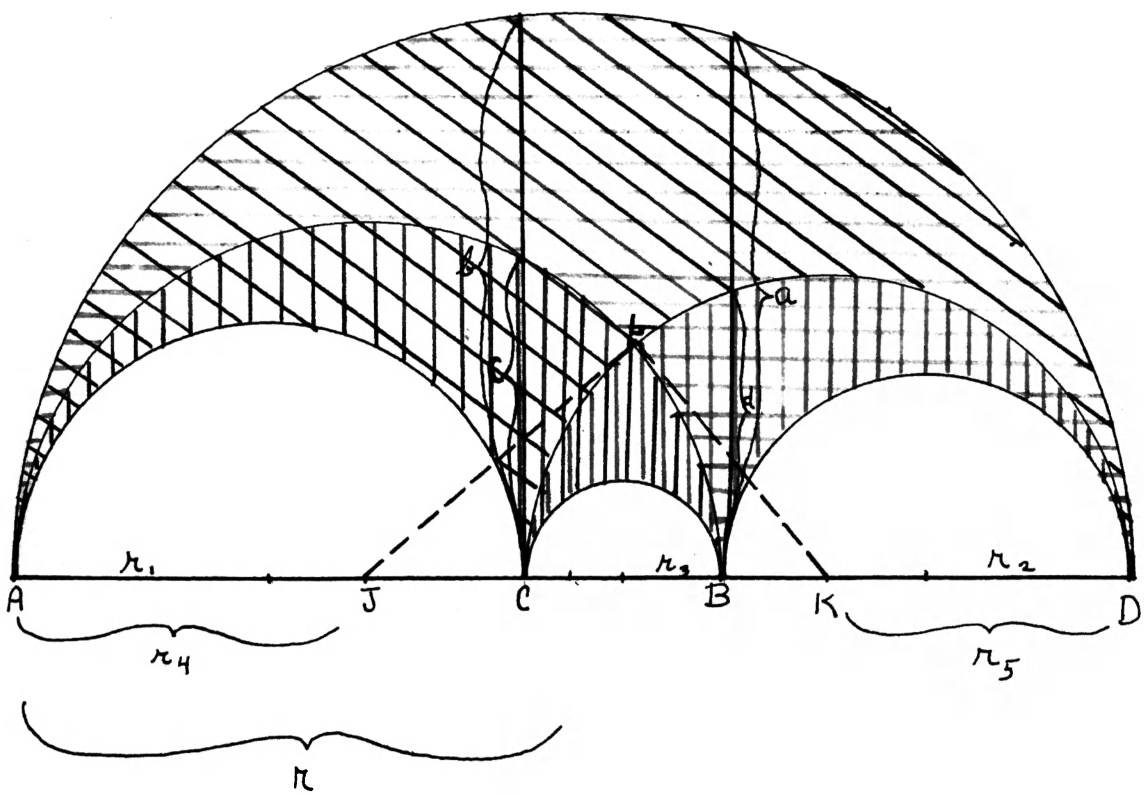

Figure 37

B. HARMONIC ARBELOI. (Figure 37)

1. Perimeters.

Let A, C, B and D be four points on a line forming a harmonic range in which A and B are one pair of conjugates and D and C the other. Draw semi-circles on AD of radius r, AB or radius r_4, AC of radius r_1, CD of radius r_5, CB of radius r_3, and BD of radius r_2. Let a represent the length of the line from the semi-circle on AD perpendicular to the point B, b the length of the line from the same semi-circle perpendicular to the point C, d the length of the line from the semi-circle on CD perpendicular to B, and c the length of the line from the semi-circle on AB perpendicular to the point C. Let I represent arbelos ABC, II arbelos CDB, III arbelos ADB and IV arbelos ADC. Then

$$\frac{AC}{CB} = \frac{AD}{BD} \quad \text{or} \quad \frac{r_1}{r_3} = \frac{r}{r_2};$$

$$\text{and } \frac{AB}{CB} = 2\frac{AD}{CD} \quad \text{or} \quad \frac{r_4}{r_3} = \frac{2r}{r_5};$$

and by dividing the last two equations,

$$\frac{r_1}{r_4} = \frac{2r_5}{r_2}.$$

$$r = r_1 + r_2 + r_3,$$

$$r_4 = r_1 + r_3,$$

$$r_5 = r_2 + r_3,$$

$$r = r_4 + r_2,$$

$$r = r_1 + r_5$$

$$a^2 = 4\, r_4 r_2$$

$$b^2 = 4\, r_1 r_5$$

$$c^2 = 4\, r_1 r_3$$

$$d^2 = 4\, r_2 r_3$$

a. The perimeter of the harmonic arbelos formed by the semi-circles on AD, DB, BC, and CA is

$$P = \pi(r + r_1 + r_2 + r_3) = 2\pi r.$$

Therefore the perimeter is equal to the circumference of the circle on AD.

b. The perimeter of I.

$$P_{I} = \pi(r_4 + r_1 + r_3)$$
$$= 2\pi r_4$$

c. The perimeter of II.

$$P_{II} = \pi(r_5 + r_2 + r_3)$$
$$= 2\pi r_5$$

d. The perimeter of III.

$$P_{III} = \pi(r + r_2 + r_4)$$
$$= 2\pi r$$

e. The perimeter of IV.

$$P_{IV} = \pi(r + r_1 + r_5)$$
$$= 2\pi r$$

2. Areas.

a. The area of I.

$$A_I = \frac{\pi}{2}(r_4^2 - r_1^2 - r_3^2)$$
$$= \frac{\pi}{2}([r_1 + r_3]^2 - r_1^2 - r_3^2)$$
$$= \frac{\pi}{2}(2r_1 r_3) = \pi r_1 r_3$$
$$= \pi (c/2)^2$$

b. The area of II.

$$A_{II} = \frac{\pi}{2}(r_5^2 - r_2^2 - r_3^2)$$
$$= \frac{\pi}{2}([r_2 + r_3]^2 - r_2^2 - r_3^2)$$
$$= \frac{\pi}{2}(2r_2 r_3) = \pi r_2 r_3$$
$$= \pi (\tfrac{d}{2})^2$$

c. The area of III.

$$A_{III} = \frac{\pi}{2}(r^2 - r_4^2 - r_2^2)$$
$$= \frac{\pi}{2}([r_2 + r_4]^2 - r_4^2 - r_2^2)$$
$$= \frac{\pi}{2}(2r_2 r_4) = \pi r_2 r_4$$
$$= \pi (\tfrac{a}{2})^2$$

d. The area of IV

$$A_{IV} = \frac{\pi}{2}(r^2 - r_1^2 - r_5^2)$$
$$= \frac{\pi}{2}([r_1 + r_5]^2 - r_1^2 - r_5^2)$$
$$= \frac{\pi}{2}(2r_1 r_5) = \pi r_1 r_5$$
$$= \pi (\tfrac{b}{2})^2$$

e. The area of harmonic arbeles formed by the semi-circles on AD, DB, BC, and CA.

$$A_H = \frac{\pi}{2}(r^2 - r_2^2 - r_3^2 - r_1^2)$$

$$= \frac{\pi}{2}([r_1 + r_2 + r_3]^2 - r_2^2 - r_3^2 - r_1^2)$$

$$= \frac{\pi}{2}(2r_1r_2 + 2r_1r_3 + 2r_2r_3)$$

$$= \pi(r_1r_2 + r_1r_3 + r_2r_3)$$

Another expression for the area of this arbeles can be found by adding the areas of arbeloi I and III

$$A_H = \pi\left(\frac{c}{2}\right)^2 + \pi\left(\frac{a}{2}\right)^2$$

= area of the circle on c plus the area of the circle on a. Likewise by adding the areas of arbeloi II and IV

$$A_H = \pi\left(\frac{b}{2}\right)^2 + \pi\left(\frac{d}{2}\right)^2$$

= area of the circle on b plus the area of the circle on d.

f. Other expressions for the areas of arbeloi I, II, III and IV.

$$A_I = \pi(r_1r_2 - r_3r_5)$$

$$A_{II} = \pi(r_1r_2 - r_3r_4)$$

$$A_{III} = \pi(r_1 r_2 + r_2 r_3)$$

$$= \pi r_1 r_2 + A_{II}$$

$$A_{IV} = \pi(r_1 r_3 + r_1 r_2)$$

$$= \pi(r_1 r_2 + A_I)$$

$$\therefore A_{III} - A_{II} = A_{IV} - A_I$$

3. Area of Triangle JKL (Figure 37)

The circles on AB and CD are orthogonal to each other since a diameter of one is cut harmonically by the two circles.* Hence the triangle JKL is a right triangle and its area is $\frac{r_4 r_5}{2}$. From the formulas of article 1 of this section, this area is also equal to rr_3 and to $r_1 r_2$.

4. Centroids of the Harmonic Arbelos

a. $\bar{x} = a$

b. $\bar{y} = \frac{2(a^2 - ad - bc)}{\pi a}$

These are applications of the last section.

c. For $\bar{X}$

$$\left(\frac{\pi a^2}{2} - \frac{\pi b^2}{2} - \frac{\pi c^2}{2} - \frac{\pi d^2}{2}\right)\bar{X} =$$

$$\frac{\pi a^3}{2} - \frac{\pi b^3}{2} - \frac{\pi c^2}{2}(2b + 2d + c) - \frac{\pi d^2}{2}(2b + d)$$

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* Daus, College Geometry, p. 32.

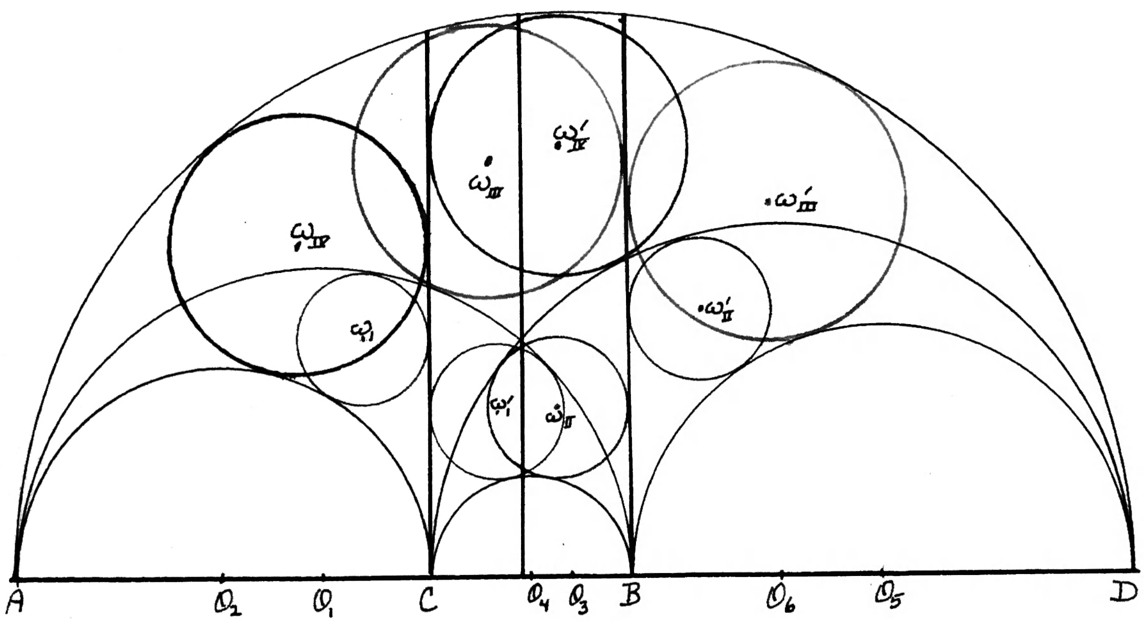
ω_{IV}
ω_1
ω_{III}
ω'_{IV}
ω'_{III}
ω'_{II}
ω'_1
ω_{II}
A
O_2
O_1
C
O_4
O_3
B
O_6
O_5
D

Figure 38

$$(a^2-b^2-c^2-d^2)\bar{X} = a^3-b^3-c^3-d^3-2bc^2-2dc^2-2bd^2$$

$$\therefore\ \bar{X} = \frac{a^3-b^3-c^3-d^3-2(bc^2+dc^2+bd^2)}{a^2-b^2-c^2-d^2}$$

d. For $\bar{Y}$

$$\frac{\pi}{2}(a^2-b^2-c^2-d^2)\bar{Y} =$$

$$\frac{\pi a^2}{2}\left(\frac{4a}{3\pi}\right)-\frac{\pi b^2}{2}\left(\frac{4b}{3\pi}\right)-\frac{\pi c^2}{2}\left(\frac{4c}{3\pi}\right)-\frac{\pi d^2}{2}\left(\frac{4d}{3\pi}\right)$$

$$(a^2-b^2-c^2-d^2)\bar{Y} = \frac{4}{3\pi}(a^3-b^3-c^3-d^3)$$

$$\bar{Y} = \frac{4(b^3+c^3+d^3+3cb^2+3cd^2+3bd^2+3bc^2+3b^2d+3c^2d+6bcd-b^3-c^3-d^3)}{3\pi\,(a^2-b^2-c^2-d^2)}$$

$$= \frac{4\cdot 3(cb^2+cd^2+bd^2+bc^2+b^2d+c^2d+2bcd)}{3\pi\,(a^2-b^2-c^2-d^2)}$$

$$= \frac{4\,[c(b+d)^2+bd(b+d)+c^2(b+d)]}{\pi\,(a^2-b^2-c^2-d^2)}$$

$$= \frac{4(b+d)[cb+cd+bd+c^2]}{\pi\,2(bc+bd+cd)}$$

$$\bar{Y} = \frac{2(b+d)(c+b)(c+d)}{\pi(bc+bd+cd)}$$

5. Inscribed Circles. (Figure 38)

a. The circles inscribed in arbelos I.

Let ρ_I be the radius, and let the centers be ω_I and ω_I'. From the formula given in Section II, Article C,1 we find that the radius has the following value,

$$\rho_I = \frac{r_1 r_3}{r_4}.$$

By the use of the formulas of part 1 of this article, the expression for ρ_I can be transformed into

$$\frac{r_1 r_3}{r_1 + r_3} .$$

The following formulas are derived in a similar way.

The circles inscribed in arbelos II. Let ρ_{II} be the radius, and the centers ω_{II} and ω_{II}'.

$$\rho_{II} = \frac{r_2 r_3}{r_5}$$

$$= \frac{r_2 r_3}{r_2 + r_3} .$$

The circles inscribed in arbelos III. Let ρ_{III} be the radius, and the centers ω_{III} and ω_{III}'.

$$\rho_{III} = \frac{r_2 r_4}{r}$$

$$= \frac{r_2 r_4}{r_2 + r_4} .$$

The circles inscribed in arbelos IV. Let ρ_{IV} be the radius, and the centers ω_{IV} and ω_{IV}'.

$$\rho_{IV} = \frac{r_1 r_5}{r}$$

$$= \frac{r_1 r_5}{r_1 + r_5} .$$

b. Relations among the inscribed circles.

(Figure 38)

The following relations involving ρ_I, ρ_{II}, ρ_{III} and ρ_{IV} can be established by using the formulas of part 1 of this article, i.e.

$$\frac{r_1}{r_3} = \frac{r}{r_2}, \quad \frac{r_4}{r_3} = \frac{2r}{r_5} \quad \text{and} \quad \frac{r_1}{r_4} = \frac{2r_5}{r_2}.$$

1. $\rho_I = \dfrac{r_1 r_3}{r_4} = r_1\left(\dfrac{r_5}{2r}\right) = \dfrac{1}{2}\,\dfrac{r_1 r_5}{r} = \dfrac{1}{2}\rho_{IV}$

2. $\rho_{II} = \dfrac{r_2 r_3}{r_5} = r_2\left(\dfrac{r_4}{2r}\right) = \dfrac{1}{2}\,\dfrac{r_2 r_4}{r} = \dfrac{1}{2}\rho_{III}$

3. $\dfrac{\rho_I}{\rho_{II}} = \dfrac{r_1 r_3}{r_4} \div \dfrac{r_2 r_3}{r_5} = \dfrac{r_1}{r_4}\cdot\dfrac{r_5}{r_2} = \dfrac{2r_5}{r_2}\cdot\dfrac{r_5}{r_2}$

$$= 2\left(\frac{r_5}{r_2}\right)^2 \text{ or } \frac{1}{2}\left(\frac{r_1}{r_4}\right)^2.$$

4. $\dfrac{\rho_{III}}{\rho_{IV}} = \dfrac{\dfrac{r_2 r_4}{r}}{\dfrac{r_1 r_5}{r}} = \dfrac{r_2}{r_5}\cdot\dfrac{r_4}{r_1} = 2\left(\dfrac{r_4}{r_1}\right)^2 \text{ or } \dfrac{1}{2}\left(\dfrac{r_2}{r_5}\right)^2$

5. $\dfrac{\rho_I}{\rho_{II}} = \dfrac{\rho_{IV}}{\rho_{III}}.$

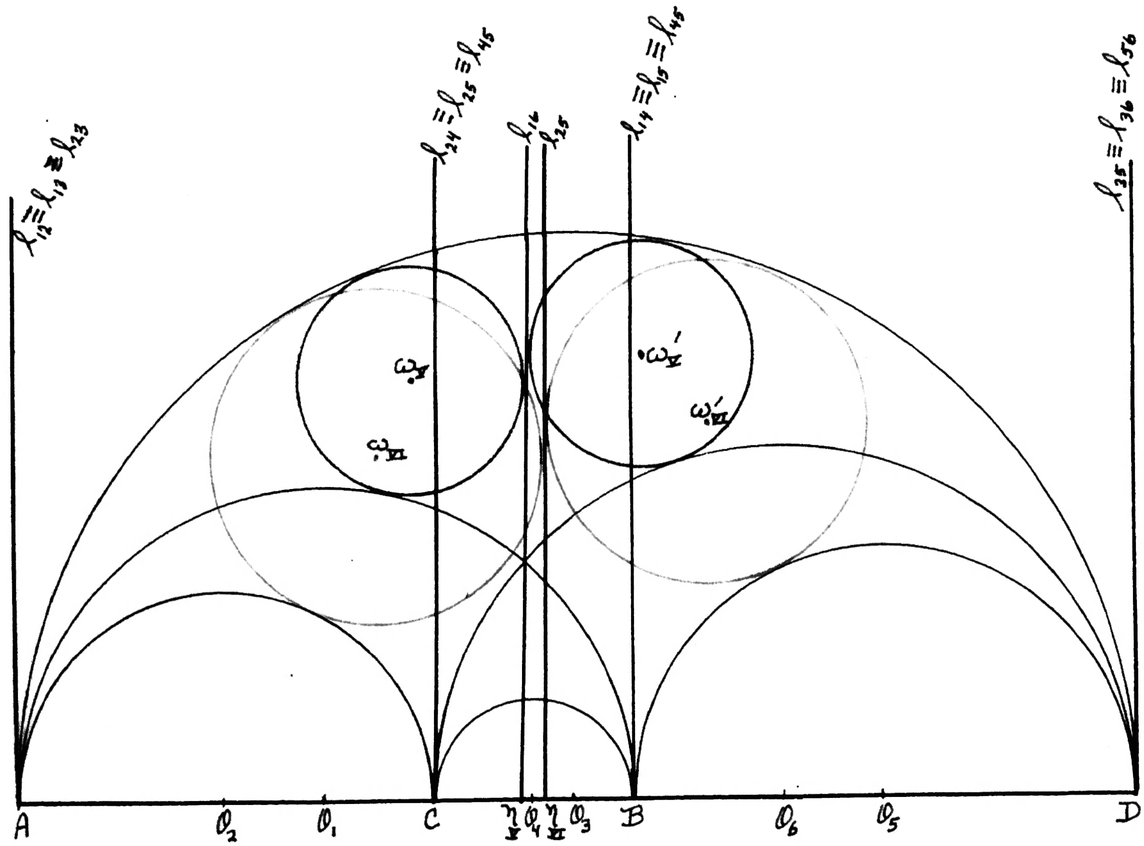
$l_{12} \equiv l_{13} \equiv l_{23}$
$l_{24} \equiv l_{25} \equiv l_{45}$
l_{16}
l_{25}
$l_{14} \equiv l_{15} \equiv l_{45}$
$l_{35} \equiv l_{36} \equiv l_{56}$
ω_V
ω_{VI}
ω_V'
ω_{VI}'
A
O_2
O_1
C
η_V
O_4
η_{VI}
O_3
B
O_6
O_5
D

Figure 39

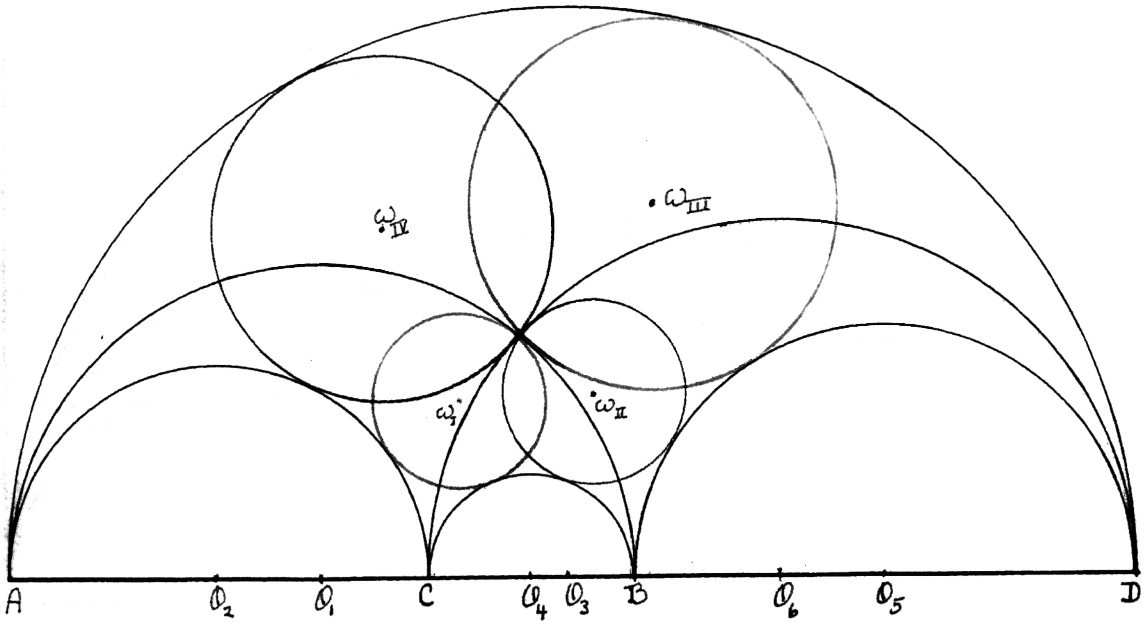

Figure 40

Relations involving ρ_{V} and ρ_{VI} of Figure 39 have been shown in Section II. In this figure there are fifteen radical axes of the six circles. Twelve of them fall into four sets of three coincident lines, but the other three radical axes are distinct.

The circles ω_{V} and ω_{V}' are a special case of Section II, Article C, 2. Their radii, ρ_{V} and ρ_{V}', are equal to:

$$\rho_{V} = \frac{A\eta_{V} \cdot D\eta_{V}}{2AD} - \frac{A\eta_{V} \cdot B\eta_{V}}{2AD}$$

$$\rho_{V}' = \frac{B\eta_{V} \cdot A\eta_{V}}{2AD} - \frac{B\eta_{V} \cdot C\eta_{V}}{2AD}$$

$$\text{and } A\eta_{V} \cdot B\eta_{V} = B\eta_{V} \cdot C\eta_{V}$$

From Section II, Article C, 3, circles ω_{VI} and ω_{VI}' have radii, ρ_{VI} and ρ_{VI}', equal to the same as those above only

$$BD = D\eta_{VI} + B\eta_{VI}$$

$$\text{and } AC = A\eta_{VI} + C\eta_{VI}$$

c. Inscribed circles (Figure 40)

Let ω_{I} be the circle inscribed in arbelos I, ω_{II} in arbelos II, ω_{III} in arbelos III, and ω_{IV} in arbelos IV. From the formula for the radius of the inscribed circle derived by V. Thebault* we find the following formulas for the radii, ρ_i , of these inscribed circles:

$$\frac{1}{\rho_I} = \frac{1}{r_1} + \frac{1}{r_3} - \frac{1}{r_4}$$

$$\frac{1}{\rho_{II}} = \frac{1}{r_3} + \frac{1}{r_2} - \frac{1}{r_5}$$

$$\frac{1}{\rho_{III}} = \frac{1}{r_4} + \frac{1}{r_2} - \frac{1}{r}$$

$$\frac{1}{\rho_{IV}} = \frac{1}{r_1} + \frac{1}{r_5} - \frac{1}{r}$$

These formulas can be changed by the use of the relations of Article 1 of this section into the following expressions:

$$\frac{1}{\rho_I} = \frac{r_3 + r_1}{r_1 r_3} - \frac{1}{r_4} = \frac{r_4}{r_1 r_3} - \frac{1}{r_4}$$

$$\frac{1}{\rho_{II}} = \frac{r_2 + r_3}{r_2 r_3} - \frac{1}{r_5} = \frac{r_5}{r_2 r_3} - \frac{1}{r_5}$$

$$\frac{1}{\rho_{III}} = \frac{r_2 + r_4}{r_2 r_4} - \frac{1}{r} = \frac{r}{r_2 r_4} - \frac{1}{r}$$

$$\frac{1}{\rho_{IV}} = \frac{r_5 + r_1}{r_1 r_5} - \frac{1}{r} = \frac{r}{r_1 r_5} - \frac{1}{r} .$$

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* Thebault, Bulletin de la Societe Mathematique, 1944. pp. 68-

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